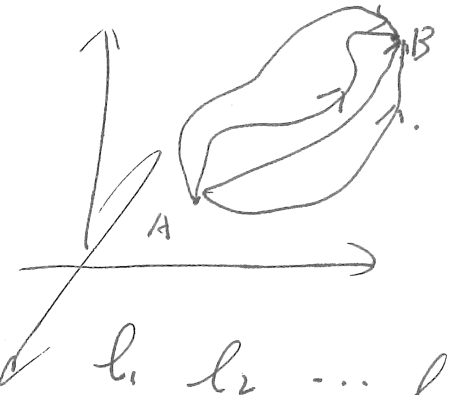
SPUM - pre-lecture. (classical mechanics)

Course idea: To understand the basic concept of classical mechanics and the fundamental ~~xxxx~~ ~~principle~~ mathematics methods of physics. (PDE, ODE, functional, variation, etc.)

1. motion of object.

Consider an object in a configuration space^{+} with all field is conserved. It ~~xxx~~ move ~~from~~



point $A \to B$. under a time interval $[t_1, t_2]$. The trajectory of the motion can be described by some curve line, $l_1\ l_2 \dots l_n$.. So, How can we know which curve gives ~~up~~ us the least action? or say, ~~do~~ to total ~~virtual~~ effective forces do the least effective work?

$$W_{eff} = \sum_i F_{e,i} \Delta x_i$$

We can use an ~~tht~~ functional to estimate ~~this~~ it.

$$\text{minimize} \to S = \int_{t_1}^{t_2} \mathcal{L}\, dt \Leftrightarrow \int_{t_1}^{t_2} W_{eff}\, dt$$

where $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$. called Lagrangian ~~Lagragian~~.

2. principle of least action.

So. how to evaluate this functional?

~~A~~ we could take the variation of it

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} \delta \mathcal{L}\, dt \;(=)\; \int_{t_1}^{t_2} \delta W_{eff}\, dt.$$

Definition, if $\delta S = 0$. then. Lagrangian ~~this solution of this~~ describes the motion with the least action. ~~($\delta \mathcal{L} = 0$)~~.

And the $\mathcal{L}$ form a PDE. called Euler-Lagrange Eq.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q}. \quad \sim \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

where $\mathcal{L} \equiv T - V$.

T: kinetic, V: potential (energy).

0. Mathematical ~~Mathematical~~ ~~foundation~~ foundation.

~~#~~ Variation.

$$\delta f(q_1, q_2, \cdots q_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \delta q_i$$

$$\delta \int \mathcal{L}^{(q_1, q_2 \cdots)} dt = \int \delta \mathcal{L}\, dt = \int \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i\, dt.$$

~~multivariables~~
multivariables. funtional .

$$\delta \int \mathcal{L}(q_1, q_2 \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots \dot{q}_n, t)\, dt$$

$$= \int \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \quad dt .$$

prove:
Consider two funtional. $F(q, \dot{q})$ $\hat{F}(\hat{q}, \dot{\hat{q}})$.
defined as

$$\begin{cases} F(q, \dot{q}) = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt \\ \hat{F}(\hat{q}, \dot{\hat{q}}) = \int_{t_1}^{t_2} \mathcal{L}(\hat{q}, \dot{\hat{q}}, t)\, dt . \end{cases}$$

~~where $\hat{F}$ is another line of trajectory that.~~
For two diffent ~~path~~ ~~Conditions.~~ path. $(q, \dot{q})$. $(\hat{q}, \dot{\hat{q}})$.

then,

$$F(\hat{q}, \dot{\hat{q}}) - F(q, \dot{q}) = \int_{t_1}^{t_2} \mathcal{L}(t, \hat{q}, \dot{\hat{q}}) - \mathcal{L}(t, q, \dot{q})\, dt .$$

$$= \int_{t_1}^{t_2} dt \Big\{ \Big[\mathcal{L}(t, \hat{q}, \dot{\hat{q}}) - \mathcal{L}(t, q, \dot{q}) \Big] + \Big\{ \frac{\partial \mathcal{L}}{\partial \hat{q}} (\hat{q} - \hat{q}) + \frac{\partial \mathcal{L}}{\partial \dot{\hat{q}}} (\dot{\hat{q}} - \dot{\hat{q}}) - \frac{\partial \mathcal{L}}{\partial q} (q - \hat{q})$$
$$- \frac{\partial \mathcal{L}}{\partial \dot{q}} (\dot{q} - \dot{\hat{q}}) \Big] + \frac{1}{2} \Big[\frac{\partial^2 \mathcal{L}}{\partial \hat{q}^2} (\hat{q} - \hat{q})^2 + \frac{\partial^2 \mathcal{L}}{\partial \hat{q} \partial \dot{\hat{q}}} (\hat{q} - \hat{q})(\dot{\hat{q}} - \dot{\hat{q}}) + \frac{\partial^2 \mathcal{L}}{\partial \dot{\hat{q}}^2} (\dot{\hat{q}} - \dot{\hat{q}})^2 - \frac{\partial^2 \mathcal{L}}{\partial q^2} (q - \hat{q})^2$$
$$- \frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}} (q - \hat{q})(\dot{q} - \dot{\hat{q}}) - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} (\dot{q} - \dot{\hat{q}})^2 \Big] + \cdots \Big\}\, dt .$$

3,

~~we~~ denote: $\begin{cases} -q+\hat{q} = \delta q \\ (-q+\hat{q})^2 = \delta q^2 \end{cases}$ $\quad \begin{cases} -\dot{q}+\hat{\dot{q}} = \delta \dot{q} \\ (-\dot{q}+\hat{\dot{q}})^2 = \delta \dot{q}^2 \end{cases}$

and $F(\hat{q},\hat{\dot{q}}) - F(q,\dot{q}) = \delta F + \delta^2 F + \dots$

So. we get

1^{st} $\quad \delta F = \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \, dt.$

2^{nd}: $\quad \delta^2 F = \int_{t_1}^{t_2} \frac{\partial^2 \mathcal{L}}{\partial q^2}\delta q^2 + \frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}} \delta q \delta \dot{q} + \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}\delta \dot{q}^2 \, dt.$

§ E-L eq of motion.

q_i is the general coordinates.

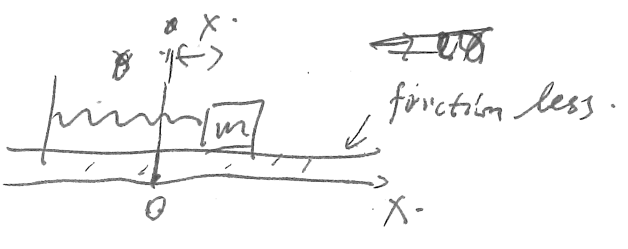
$\dot{q}_i = \frac{dq_i}{dt}$ is the general velocity.

define $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$ called general momentum.

the equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

Ex1. Harmonic oscillator.



~~$\mathcal{L} = \frac{1}{2}m\dot{q}^2$~~

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

~~for~~ EOM. (equation of motion). is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \iff m\ddot{x} = -kx$$

$$\overline{m\vec{a}} = \overline{\vec{F}_{total}}.$$

$m\ddot{x} = -kx$ define $w = \sqrt{\frac{k}{m}}$.

$\Rightarrow \ddot{x} + w^2 x = 0$

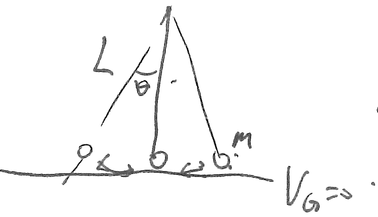
characteristic eq. $\lambda^2 + w^2 = 0$

$\Rightarrow \lambda = \pm i w$.

So. $x(t) = C_1 \sin wt + C_2 \cos wt$.

where. C_1, C_2 are some constant.

Ex2. ~~pendulum~~ single pendulum.



$\mathcal{L} = \frac{1}{2} I \dot{\theta}^2 - mgL\cos\theta$.

E-L. eq. should be.

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}$, $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\ddot{\theta} = mL^2\ddot{\theta}$

$\frac{\partial \mathcal{L}}{\partial \theta} = mgL\sin\theta$.

$\Rightarrow mL^2\ddot{\theta} = mgL\sin\theta$

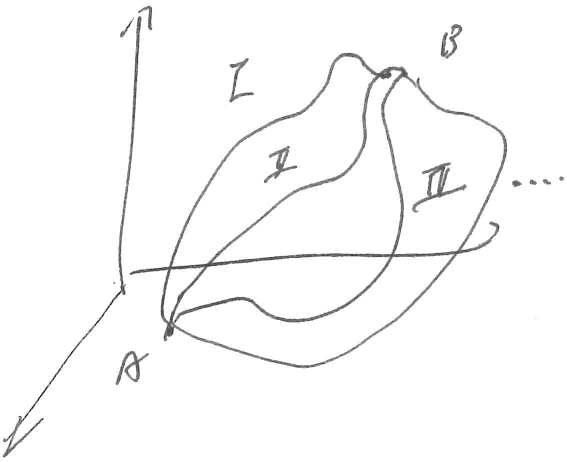
$\Rightarrow \ddot{\theta} = \frac{g}{L}\sin\theta$.

For small damping. $\sin\theta \approx \theta$.

$\ddot{\theta} = \frac{g}{L}\theta \Rightarrow \theta = C_1 e^{\sqrt{\frac{g}{L}}t} + C_2 e^{-\sqrt{\frac{g}{L}}t}$.

- virtual ~~force &~~ work.

Def: The work of a force ~~acting~~ on a particle as it moves along a displacement is different ~~from~~ for different displacement. ~~the~~ All possible displacements of a particle may follow are called virtual displacement. ~~all~~ The work of a force on a particle along a virtual displacement is known as the virtual work.



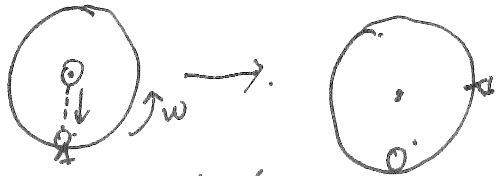
I, II, III: virtual displacements.

on force F_i along work ~~do~~ along it

$$W = \int_{I,\ or\ II,\ or\ III} \vec{F}_i \cdot d\vec{l} = \sum_i \vec{F}_i \cdot \Delta \vec{q}_i$$

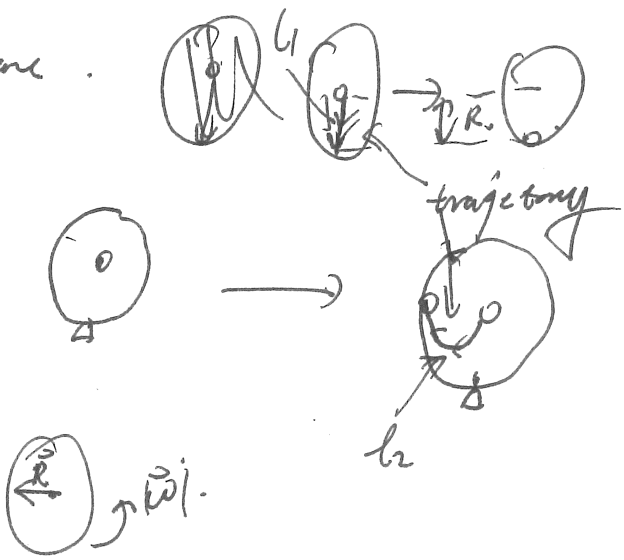
$\delta W = \sum_i \vec{F}_i \delta \vec{q}_i$ ← virtual force ~~difference~~ difference.

EX1:



In ~~enforce~~ ~~stationary~~ inertial reference frame.

In rotating ~~the~~ reference frame



~~The~~ ~~the~~ force in ~~stationary~~ internal frame.

$$F_I = mg$$

In rotation frame.

$$F_R = mg - m\frac{d\vec{\omega}}{dt}\times\vec{R} - 2m\vec{\omega}\times\vec{v} - m\vec{\omega}\times(\vec{\omega}\times\vec{R})$$

$= ~~m~~mg - F_{Euler} - F_{coriolis} - F_{centrifugal}$

~~R~~ Notice

$$\int_{l_1} \vec{F}_I \cdot d\vec{l} \doteq \int_{l_2} \vec{F}_R \cdot d\vec{l} \doteq W. \quad \text{and } (\delta W = 0)$$

~~The~~ That means, the total work that drawn by the fictitious force ~~one~~ is zero. The total virtual work that in the two system are the same. (~~the invariant~~ invariant.)

(X If. The total virtual ~~Work~~ work ~~drawn~~ done by the effective force ~~goes to~~ ~~to~~ is zero ~~minimum~~. $(W_e = 0)$ ~~$(\delta W_e = 0)$~~
Then, the ~~that~~ the system is in a dynamic equilibrium.
~~The, the~~)

— D'Alembert's principle.

Def: ~~the p~~ The sum of the differences between the forces acting on a system of massive particles and the time derivatives of the momenta of the system itself projected ~~ed~~ onto any virtual displacement within the constrain of the system is zero.

$$\sum_{i=1}^{N} (\underbrace{\vec{F}_i}_{\text{For force and inertia.}} - m_i \underbrace{\vec{a}_i}_{\text{acceleration.}}) \delta R_i = 0 \Leftrightarrow \sum_{i=1}^{N} (\vec{F}_i - \dot{P}_i) \delta R_i = 0$$

with $m_i \vec{a}_i = \dot{P}_i$.

Notice

$$\sum (F_i - m a_i) \cdot \delta R_i = \delta W = 0$$

That's ~~the~~ leads to the principle of virtual ~~wake~~ work for applied force.

- To the least action princ.

Remember that

$$\delta S = \int_{t_1}^{t_2} \delta W \, dt \quad \text{if } \delta W = 0.$$

then $\delta S = 0$. that's lead us back to the principle of least action.