SPUM. Lecture ~~two~~ 2.

- Conservation law.

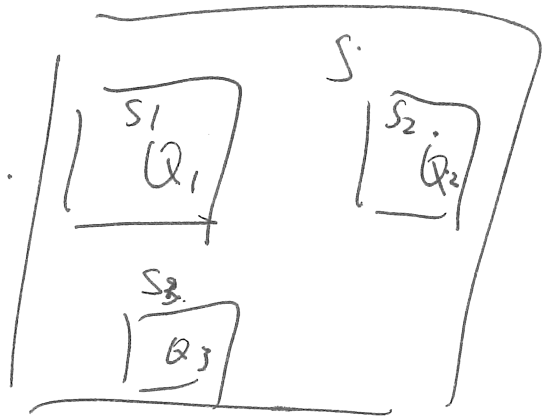
* Integral of the motion.

Def: In a mechanical system, a ~~fution~~ function of these quantities $(q_i, \dot{q}_i)$ remain constant during the motion, and depend only on the initial condition. such funtions are called ~.

The importance of the homogeneous and ~~isotp~~ isotropic ~~of space~~ of space is a kind of symmetry. this kind of symmetry is ~~meaningfull~~ meaningfull.

This kind of integral of motion ~~the~~ are called conserved quanties.

properties: additable.



$$Q_S = \cancel{S_1 +} Q_1 + Q_2\ Q_3$$

* ~~Erom~~ Energy conservation. (In a closed system)

From the homogenous of time → conservation of Energy.

$$\frac{dL(q,\dot{q})}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

Using $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$ ~~rependeniy~~ dt.

$$\Rightarrow \frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right)}_{\parallel\ E} = 0$$

E is ~~some~~ a ~~conservati~~ conserved quantity.

Therefore, for motions in ~~sec~~ colosed systems, a ~~M~~ IoM.

is an invariant called energy of the system

also called ~~conserat~~ conserved system.

Notice: this law not only true for closed system. if ~~e~~ external field is time independent.

For closed system

$$L = T(q, \dot{q}) - U(q)$$

as $T = \sum_i \frac{1}{2} m \dot{q}_i^2$

Notice $\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 \cdot \sum \frac{1}{2} m \dot{q}_i^2 = 2T$.

$\Rightarrow E = T + U$.

omentum . $\vec{p}$ ~ momenta . ~~f~~ momt = ~~toque~~ torque

This Q. is due to the homogenous of space.

For a infinitesmal ~~ch.~~ traslational motion. in a closed system:

$$\vec{r} \to \vec{r} + \vec{\varepsilon}$$

$$\delta \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial \vec{r}_i} \cdot \delta \vec{r}_i = \varepsilon \cdot \sum_i \frac{\partial \mathcal{L}}{\partial r_i} = 0.$$

~~From least action principle~~.

$$\cancel{\delta S = \int \delta \mathcal{L} \, dt = 0}$$

For all mass point, they ~~are~~ move together. the $\mathcal{L}$ shou not be changed $\Leftrightarrow \delta \mathcal{L} = 0$.

$$\Rightarrow \sum_i \frac{\partial \mathcal{L}}{\partial r_i} = 0.$$

$$\text{as} \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}_i} = \frac{\partial \mathcal{L}}{\partial r_i}$$

$$\Rightarrow \frac{d}{dt}\left(\sum \frac{\mathcal{L}}{\partial \dot{r}_i} \right) = 0$$

$$\Rightarrow Q = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{r}_i}.$$

$$P \equiv \sum_i \frac{\partial \mathcal{L}}{\partial \dot{r}_i} \qquad \text{as } \mathcal{L} = \sum \tfrac{1}{2} m_i \dot{r}_i^2 + U(q)$$

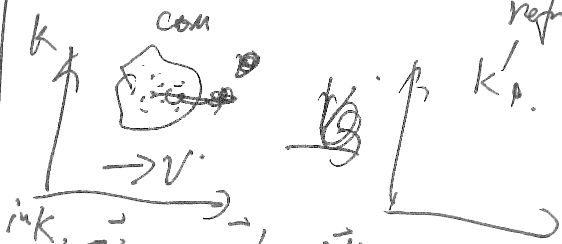
$$\Rightarrow \vec{P} \equiv \sum_i m_i \vec{\dot{r}}_i.$$

For general coordinates.

$$\begin{cases} \vec{p}_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ \vec{F}_i = \dfrac{\partial \mathcal{L}}{\partial q_i} \end{cases}$$

we have $\boxed{\dfrac{d}{dt}\vec{p}_i = \vec{F}_i}$

COM. in different reference frame.



in K. $\vec{v}_i = \vec{v}_i' + \vec{V}$

$$E = \frac{1}{2}\sum_i m_i v_i^2 + U = \frac{1}{2}\sum_i m_i(\vec{v}_i' + \vec{V})^2 + U$$

$$= \frac{MV^2}{2} + \underbrace{\vec{V}\cdot\sum_i m_i \vec{v}_i'}_{\vec{V}\cdot\vec{P}} + \underbrace{E'}_{\text{in } K'}$$

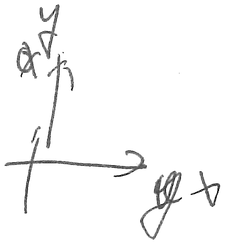
~~* Angular momentum~~

* symmetry related to the conservation law.

we know that if $\mathcal{L}(q_i, \dot{q}_i)$ then

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

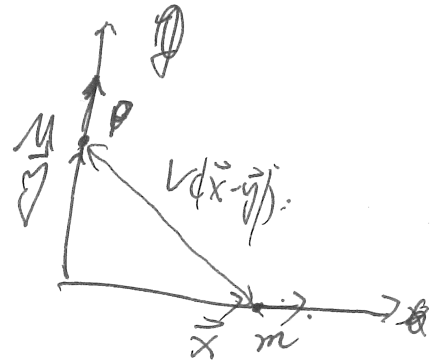
Consider a 2-D ~~cartesian~~



$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - V(x-y)$$

two particles. $|\vec{x} - \vec{y}| = \vec{u}$

m. M.



By ~~chang~~ changing variables

$$\begin{matrix} \sqrt{m}\,\vec{x} = \vec{q}_1 \\ -\sqrt{M}\,\vec{y} = \vec{q}_2 \end{matrix} \Rightarrow \begin{cases} \vec{x} = \dfrac{\vec{q}_1}{\sqrt{m}} \\ \vec{y} = -\dfrac{\vec{q}_2}{\sqrt{M}} \end{cases} \text{ and } \begin{cases} \dot{\vec{x}} = \dfrac{\dot{\vec{q}}_1}{\sqrt{m}} \\ \dot{\vec{y}} = -\dfrac{\dot{\vec{q}}_2}{\sqrt{M}} \end{cases}$$

Then $\mathcal{L}$ becomes.

$$\mathcal{L} = \frac{1}{2}(\dot q_1^2 + \dot q_2^2) - V\left(\left|\frac{q_1}{\sqrt m} + \frac{q_2}{\sqrt M}\right|\right)$$

The E-L equation leads.

$$\frac{\partial \mathcal{L}}{\partial q_1} = -\frac{1}{\sqrt m} V'\left(\frac{q_1}{\sqrt m} + \frac{q_2}{\sqrt M}\right) = \dot p_1 = \cancel{\frac{d}{dt}}$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = -\frac{1}{\sqrt M} V'\left(\frac{q_1}{\sqrt m} + \frac{q_2}{\sqrt M}\right) = \dot p_2$$

$$\Rightarrow \left(\frac{\dot p_1}{\sqrt M} - \frac{\dot p_2}{\sqrt m}\right) = \frac{d}{dt}\left(\frac{p_1}{\sqrt M} - \frac{p_2}{\sqrt m}\right) = 0 \ !$$

Moreover. if $\mathcal{L} = T - V(aq_1 + bq_2)$

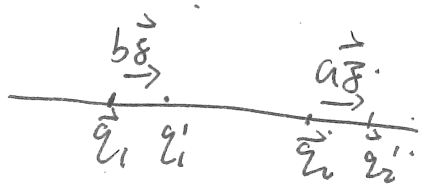
$$\Rightarrow \frac{d}{dt}(bp_1 - ap_2) = 0$$

or $Q = bp_1 - ap_2$ is conserved.

$(q, \dot q) \xrightarrow{a} \cancel{(Q, \dot Q)}\ (q', \dot q')$. and $\mathcal{L} = T - V(aq_1 + bq_2)$

~~Define~~: If $\delta\mathcal{L} = 0$. Then U is symmetric transformation.

For two particle $(q_1, \dot q_1)$, $(q_2, \dot q_2)$. move $b\xi$, $a\xi$ simultaneously



$$\begin{cases} q_1' = q_1 + b\xi \\ q_2 = q_2 + a\xi \end{cases}$$

$\Rightarrow T = T'$ $\Leftrightarrow \delta T = 0$

$V' = V(aq_1' + bq_2') = V(aq_1 + ab\xi + bq_2 - ab\xi)$

$= V(aq_1 + bq_2) = V$

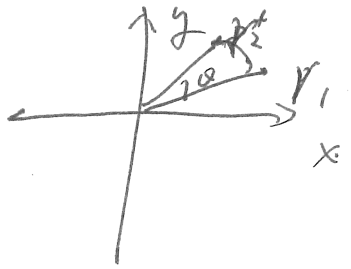
$\Leftrightarrow \delta V = 0$

$\Rightarrow \delta\mathcal{L} = 0$

This is translational symmetry.

- Angular motion.

From the rotational symmetric..



$$\mathcal{L} = \frac{m}{2}[\dot{x}^2 + \dot{y}^2] - V(x^2+y^2)$$

applying a $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

one Basis ~~(x, y)~~ $r_1 = \begin{bmatrix} x \\ y \end{bmatrix}$. $r_2 = \begin{bmatrix} x' \\ y' \end{bmatrix}$

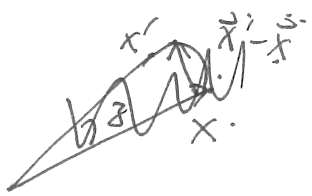
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

if θ is small, ~~let θ~~ $\theta^2 = 0$. ~~and~~ $\delta^2 = 0$.

$\Rightarrow \begin{cases} \sin\theta = \theta \\ \cos\theta = 1 \end{cases}$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} x' = x + \theta y \\ y' = y - \theta x \end{cases}$$

or. $\begin{cases} x' - x = \theta y = \delta x \\ y' - y = -\theta x = \delta y. \end{cases}$



$$\begin{cases} \frac{d}{dt}(x' - x) = \frac{d}{dt}(\delta x) = \delta \dot{x} = \theta \dot{y} \\ \frac{d}{dt}(y' - y) = \frac{d}{dt}(\delta y) = \delta \dot{y} = -\theta \dot{x}. \end{cases}$$

using $\delta(x^2+y^2) = 2x\delta x + 2y\delta y$.

replacing it by what we have

$$\Rightarrow \delta(x^2+y^2) = 2\theta xy - 2\theta xy = 0$$

$$\Rightarrow \delta V(x^2+y^2) = 0$$

$$\delta(\dot{x}^2+\dot{y}^2) = 2\dot{x}\delta\dot{x} + 2\dot{y}\delta\dot{y} = 0.$$

$$\Rightarrow \delta T = 0.$$

$\Rightarrow \delta \mathcal{L} = 0$. this transformation $R(\theta)$ is symmetric

- Generally. $\begin{cases} \delta q_i = f_i(q_i)\cdot\delta t, & \delta t \to 0 \\ \delta\dot{q}_i = \frac{d}{dt}(\delta q_i) \end{cases}$

the variation of $\mathcal{L}$ is

$$\delta\mathcal{L}(\dot{q}_i, q_i) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i \right)$$

$$= \sum_i \left(\dot{p}_i \delta q_i + p_i \delta\dot{q}_i \right)$$

$$= \sum_i \frac{d}{dt}(p_i\cdot\delta q_i)$$

$$= \frac{d}{dt}\left(\sum_i p_i f_i(q_i)\cdot\delta t\right) = 0 \quad \text{if it is symmetric}$$

~~on the other hand~~

EX: if $\frac{d}{dt}\left[\sum_i p_i f_i(q)\right]=0$

let $Q=\sum_i p_i f_i(q)$

Using the translational model.

$$\begin{cases} \delta q_1 = b\cdot \delta t \\ \delta q_2 = -a\cdot \delta t \end{cases} \Rightarrow \begin{cases} f_1 = b \\ f_2 = -a \end{cases}$$

let $Q = bp_1 - ap_2$, $\frac{d}{dt}Q = \frac{d}{dt}(bp_1 - ap_2) = 0$

EX.. for rotation:

$$\begin{cases} \delta x = y\cdot\theta \\ \delta y = -x\cdot\theta \end{cases} \Rightarrow \begin{cases} f_x = y \\ f_y = -x \end{cases}$$

$Q = yp_x - xp_y \Leftrightarrow \frac{d}{dt}(\underbrace{yp_x - xp_y}_{= L_z})=0$

L_z : angular momentum.