PSUM . 101   lecture 3.

- Integration of the equation of motion.

Sometimes, for a system in fixed external condition, we no need to solve the equation of motion, i.e. directly get the relation between position ($\vec{r}$) and time (t): ~~instead~~ of the equation motion. method by using integration

Usually, we starting formulation from the conservation of total energy (For a closed, isolated system)

~~the~~ 1-D system.

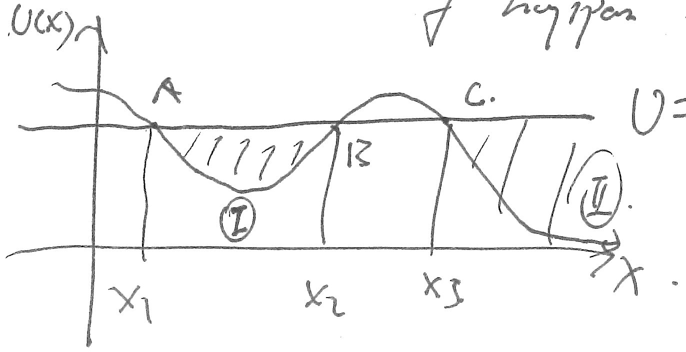
$$E = \frac{1}{2} m \dot{x}^2 + U(x).$$

$$\Rightarrow \quad t = \sqrt{\frac{m}{2}} \int dx \frac{1}{\sqrt{E - U(x)}} + \text{costant}.$$

we get the solution of this system directly.

For classical motion. ~~$E \geq 0$~~ ~~and~~ $E > U(x)$

So, the motion can only happen in the region where $E > U(x)$



(I): the motion is called "finite". (1-D case is called oscillation)
Bounded.

(II): limit only on one side. "infinite" is called.

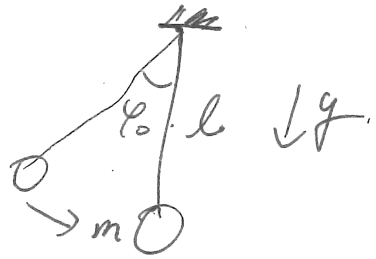
BC is called "potential well"

Between AB part, (x_1, x_2). The period of this oscillation is

$$T(E) = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}$$

$$= 2t$$

Example: ~~of pendulum~~.
Determine the period of oscillations of a simple pendulum as a function of the amplitude of the oscillation.
"$T \sim \varphi$" relation



$$E = \frac{ml^2\dot{\varphi}^2}{2} - mgl\cos\varphi = -mgl\cos\varphi_0.$$

$$T = 4\,t_{0\to\varphi_0} = 4\sqrt{\frac{l}{2g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = 2\sqrt{\frac{l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\sin^2(\frac{\varphi_0}{2}) - \sin^2\frac{\varphi}{2}}}$$

let $\sin\theta = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}$

$$\Rightarrow T = 4\sqrt{\frac{l}{g}}\,K\left(\sin\frac{\varphi_0}{2}\right).$$

where $K(k) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is called the complete elliptic integral of the first kind. 2.

of the expansion

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\varphi_0^2 + \cdots\right).$$

The first term is the period equation that we commonly used.

— reduced mass:

For single ~~mass point~~. Interacting particle problem. We can use single mass to describe it individually. For two interacting particles, ~~we need to replace the COM and~~ we could simplified the calculation of solving this system, by separating the motion of the system into the motion of centre of mass and that of particles relative to the centre of mass.

$$\mathcal{L} = \frac{m_1}{2}\dot{r}_1^2 + \frac{m_2}{2}\dot{r}_2^2 - U(|r_1 - r_2|)$$

let $r = r_1 - r_2$. let the origin at the centre of mass of these two particles.

$$\begin{cases} m_1 r_1 + m_2 r_2 = 0 = (m_1 + m_2)\, r_{com} \\ r = r_1 - r_2 \end{cases}$$

$$\Rightarrow r_1 = \frac{m_2}{m_1+m_2} r, \quad r_2 = \frac{-m_1}{m_1+m_2} r$$

$$m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

thus $\mathcal{L} = \frac{1}{2} m \dot{r}^2 - U(r)$, m is called reduced mass.

3.

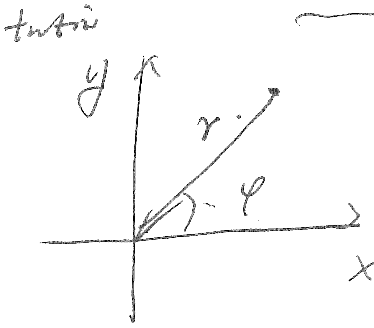
— Motion in a central field.

In last lecture. ~~we~~ we ~~first~~ have already use the central field system to derived the conservation of angular motion.

Def: The problem of determining the motion of a single particle in external field s.t. it's potential energy depends only on the distance $\vec{r}$ from some fixed pt.

i.e.

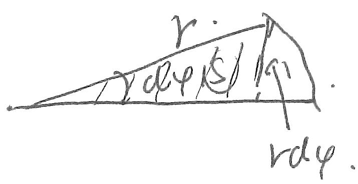
$$\underline{U(r)}. \quad , \vec{F} = -\frac{\partial U}{\partial r} = -\frac{dU}{dr}\frac{\vec{r}}{|\vec{r}|}.$$



$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r).$$

$$L_z = mr^2\dot{\varphi} = \text{contant}.$$

$$S = \frac{1}{2} r \cdot r d\varphi \equiv df \rightarrow \text{Sector area}$$



$$r^2 d\varphi = 2 df$$

$$m r^2 \frac{d\varphi}{dt} = 2m\frac{df}{dt}$$

Sector velocity

$$\Rightarrow mr^2\dot{\varphi} = 2m\dot{f} = L_z = \text{contant}$$

$\Rightarrow \dot{f} = \text{contant}$ (kepler's second law)

* ~~里~~. just from the conservation of L_z and E, we could get the integration of Equation of motion.

$$\Rightarrow E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) = \frac{m\dot{r}^2}{2} + \frac{L_z^2}{2mr^2} + U(r).$$

$$\Rightarrow \dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}[E-U(r)] - \frac{L_z^2}{m^2r^2}}$$

$$\Rightarrow t = \int dr \cdot \frac{1}{\sqrt{\frac{2}{m}[E-U(r)] - \frac{L_z^2}{m^2r^2}}} + \text{constant}. \qquad (1)$$

undetermined

as $L_z = mr^2\frac{d\varphi}{dt}$

$$\Rightarrow d\varphi = \frac{L_z}{mr^2}dt.$$

$$d\varphi = \frac{L_z}{mr^2}dt = \frac{L_z}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}[E-U(r)] - \frac{L_z^2}{m^2r^2}}}$$

$$\Rightarrow \varphi = \frac{L_z}{mr^2}\int \frac{dr}{\sqrt{\frac{2}{m}[E-U(r)] - \frac{L_z^2}{m^2r^2}}} + \text{const}. = \int \frac{\frac{M}{r^2}dr}{\sqrt{2m[E-U] - \frac{M^2}{r^2}}} + \text{const}! \qquad (2)$$

~~That~~'s the (1) (2) are solutions of the Centre force system.

— Effective potential, centrifugal energy

$$L = \frac{m}{2}\dot{r}^2 + \frac{L_z^2}{2mr^2} + U(r).$$ just see from $\vec{r}$ direction is a 1-D motion.

$$U_{eff} \equiv \underbrace{\frac{L_z^2}{2mr^2}}_{\text{centrifugal energy}} + U(r).$$

$T_r = \frac{m}{2}\dot{r}^2$

$U_{eff} = \frac{L_z^2}{2mr^2} + U$

$T_r + U_{eff} = E$

$\boxed{\dot{r} = 0} \Rightarrow$ [turning point]

Three important type. $U \propto \frac{1}{r}$, r^2. space oscillator.

— Kepler's problem

$U(r) \propto \frac{1}{r}$, gravity field. ~~Coulo~~ Coulomb's field.

1° attraction ~~field~~ force

$U = -\frac{\alpha}{r}$, $\alpha \in \mathbb{R}^+$,

$U_{eff} = -\frac{\alpha}{r} + \frac{L_z^2}{2mr^2}$

orbiting equation (2).

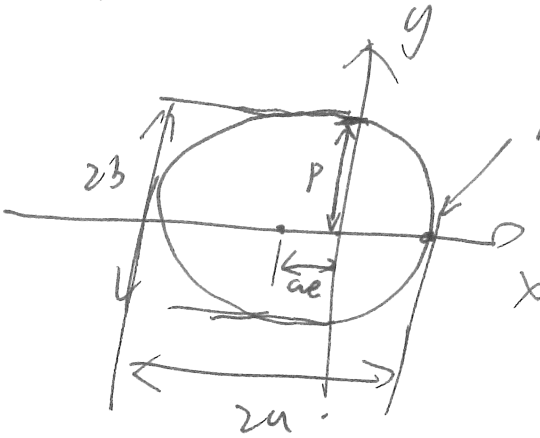
$\Rightarrow \varphi = \cos^{-1} \frac{\frac{L_z}{r} - \frac{m\alpha}{L_z}}{\sqrt{2mE + \frac{m^2\alpha^2}{L_z^2}}}$ + const. → 0.

Define $p = \frac{L_z^2}{m\alpha}$, $e = \sqrt{1 + \frac{2EL_z^2}{m\alpha^2}}$

$\Rightarrow \boxed{\frac{p}{r} = 1 + e\cos\varphi}$

~~...~~ this the second definition of ellips

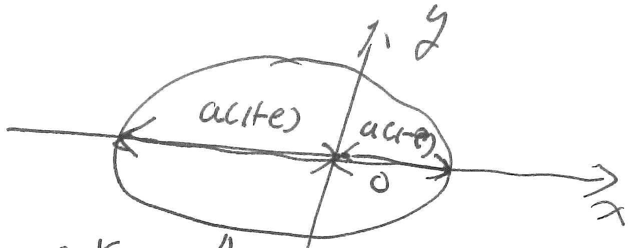
where $\begin{cases} 2p: \text{ latus rectum.} \\ e: \text{ eccentricity.} \end{cases}$



as $e<1$, $\Rightarrow E<0$, the motion is finite.

$$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}.$$

$$b = \frac{p}{\sqrt{1-e^2}} = \frac{L_z}{\sqrt{2m|E|}}$$



$$\begin{cases} r_{min} = \frac{p}{1+e} = a(1-e) \\ r_{max} = \frac{p}{1-e} = a(1+e). \end{cases}$$

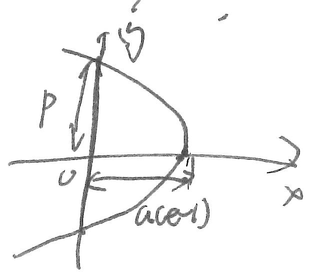
The period is given by

$$2mf = TM. \quad f = \pi ab$$

$$\Rightarrow T = 2\pi a^{\frac{3}{2}} \sqrt{\frac{m}{\alpha}} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}$$

$\boxed{T = T(E)}$ time..

For $E \geq 0$, $e \geq 1$ the orbital is ~~parabolic~~ hyperbolica..



The motion is infinity

$E=0, \quad e=1$

the orbital is a parabola. ~~of particle~~.

For ~~repulsion~~ repulsive field. the situation is the same. plz. read it via "Mechanics."