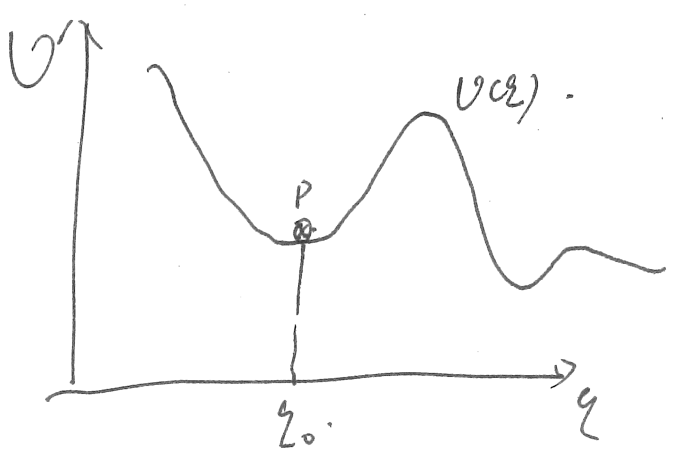
SPHM 101 Lecture 4.

Small oscillation.

— The Harmonic approximation of potential ~~en~~ energy.

In some physics ~~cal~~ problems like the oscillation of molecular, particle, ~~the os~~ These kinds of problem is really hard to be formulated / described by certain ~~phy~~ equations.
But. ~~#~~ we ~~cal~~ could using some approximation method to ~~a~~ simplify the ~~calc~~ calculation which leads to ~~#~~ a simpler result~~s~~. The most common method is Harmonic ~~approxi~~ approximation.

Consider a potential diagram $U(q)$



If particle p has a small ~~????~~ movement away from the equilibrium position $q=q_0$, ~~it will~~ a ~~a force~~ general force.
$F = -\frac{dU}{dq}$ will acting on it and force it back to the original point.

If this ~~moment~~ movement is small, we could expand $U(q)$ at $q=q_0$.

$$U(q) = U(q_0) + \frac{1}{1!} U'(q_0)(q-q_0) + \frac{1}{2!}(q-q_0)^2 \cdot U''(q_0) + O(q^3)$$

as $U(q_0) \neq 0$, $U'(q_0) = 0$, $U''(q_0) \neq 0$

$$\Rightarrow \quad U(q) \approx U(q_0) + \frac{1}{2} U''(q_0)\,(q-q_0)^2.$$

Define $U(x) = U(q) - U(q_0) = \frac{1}{2} U''(q_0)\,(q-q_0)^2$

$x = q - q_0$, $\dot{x} = \dot{q}$

we get $U(x) = \frac{1}{2} k x^2$. where. $k = U''(q_0)$.

And the kinetic energy is $T = \frac{1}{2} m \dot{q}^2 = \frac{1}{2} m \dot{x}^2$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2.$$

Using E-L equation,

$$\frac{\partial \mathcal{L}}{\partial x} = -kx, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} \quad \Rightarrow m\ddot{x} + kx = 0.$$

Define $\omega^2 = \frac{k}{m}$ $\Rightarrow$ EoM: $\ddot{x} + \omega^2 x = 0$.

The ~~solution~~ characteristic equation is given by $\lambda^2 + \omega^2 = 0$

$$\Rightarrow \lambda_1 = \omega$$
$$\lambda_2 = -\omega$$

So. the solution is

$$x = C_1 \sin(\lambda_1 x) + C_2 \cos(\lambda_2 x).$$

$$= C_1 \sin \omega x + C_2 \cos \omega x.$$

$$= \sqrt{C_1^2 + C_2^2}\,\cos(\omega t + \alpha). = A \cos(\omega t + \alpha)$$

where: $\begin{cases} \sin\alpha = \frac{-C_1}{\sqrt{C_1^2 + C_2^2}} \\ \cos\alpha = \frac{C_2}{\quad} \end{cases}$, $A = \sqrt{C_1^2 + C_2^2}$.

A : amplitude. ω: ~~angular~~ frequency (~~f~~

α: phase.

So, the total eng of ~~this~~ system hows

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{m}{2}(\dot{x}^2 + \omega^2 x^2)$$

$$= \frac{1}{2} m \omega^2 A^2.$$

$\Rightarrow E \propto A^2$. this model is called free "~~damping~~ oscillator".

— Forced ~~damping~~ oscillation

Condition. the ~~Extend~~ extend force field is large but not large enough to cause a large displacement.

An extend field is given by

$$U_e(x,t) = U_e(0,t) + x \frac{\partial U_e}{\partial x}\Big|_{x=0} + O(x^2).$$

$$\simeq U_e(0,t) + x \frac{\partial U_e}{\partial x}\Big|_{x=0}.$$

as $U_e(0,t)$ is independ with position and velocity. it can be omitted in E-L equation. ~~So, our~~. let $F(t) = -\frac{\partial U_e}{\partial x}\Big|_{x=0}$

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 \cancel{+ x F(t)} - x\frac{\partial U_e}{\partial x}\Big|_{x=0} - U_e(0,t) = \frac{1}{2}m\dot{x}^2 - \frac{k}{2}x^2 + x F(t) - U_e(0,t).$$

E-L eqution becomes.

$$\boxed{\ddot{x} + \omega^2 x = \frac{1}{m} F(t)}$$

Second-order ~~linear~~ nonhomogenous equation.

* A meaningful example

AS $F(t) = f\cos(\gamma t+\beta)$.

EoM ~~Et~~ ~~equation~~ becomes

$$\ddot{x}^2 + \omega^2 x = f\cos(\gamma t+\beta)$$

the solution should be conducted by two part

$$x = \underline{x_{homo}} + x_{non\text{-}homo}.$$

$$x_{homo} = A\cos(\omega t+\alpha).$$

Assuming $x_{non\text{-}homo} = f\left(C_3\cos(\gamma t+\beta) + C_4\sin(\gamma t+\beta)\right)$.

$$\ddot{x}_{non\text{-}homo} = f\left(C_3\gamma^2\cos(\gamma t+\beta) - C_4\gamma^2\sin(\gamma t+\beta)\right)$$

$$\Rightarrow C_3(\omega^2-\gamma^2)\cos(\gamma t+\beta) + C_4(\omega^2-\gamma^2)\sin(\gamma t+\beta) = \frac{1}{m}\cos(\gamma t+\beta)$$

$$\Rightarrow \begin{cases} C_3(\omega^2-\gamma^2) = \frac{1}{m} \\ C_4(\omega^2-\gamma^2) = 0 \end{cases} \Rightarrow \begin{cases} \omega \neq \gamma. \\ C_3 = \frac{1}{m(\omega^2-\gamma^2)} \\ C_4 = 0. \end{cases}$$

$\Rightarrow$ when $\omega \neq \gamma$.

$$x = A\cos(\omega t+\alpha) + \frac{f}{m(\omega^2-\gamma^2)}\cos(\gamma t+\beta).$$

For ~~resonance~~ resonance case

$$\ddot{x} + \omega^2 x = \frac{f}{m}\cos(\omega t+\beta).$$

The solution is given by $x = t\left(C_5\cos(\omega t+\beta) + C_6\sin(\omega t+\beta)\right)$

$c_5 = 0$, $C_6 = \frac{f}{2m\omega}$.

$x_{\text{in-hom}} = \frac{f}{2m\omega} t \sin(\omega t + \beta)$

divergence.

$\Rightarrow x = A \cos(\omega t + \alpha) + \frac{f}{2m\omega} t \sin(\omega t + \beta)$

— Beats

In some ~~osc~~ oscillation, the amplitude of oscillation will ~~of~~ change along some period "ε". in some range. This ~~phenomen~~ phenomenon is called "beats".

Starting from the oscillation ~~of~~ with external forced ~~fr~~ frequency $\gamma \approx \omega$. forced : $\gamma = \omega + \varepsilon$. $\varepsilon \to 0$.

the solution in complex form is given by.

$$x = \underbrace{A e^{i\omega t}}_{x_{\text{homo}}} + \underbrace{B e^{i\gamma t}}_{x_{\text{in-homo}}} \Leftrightarrow A\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta).$$

$$x = \cancel{(A + e^{i\varepsilon t}}$$

$$A e^{i\omega t} + B e^{i(\omega + \varepsilon)t} = (A + Be^{i\varepsilon t}) e^{i\omega t}.$$

The amplitude. "C" is defined as

$$C = \sqrt{(A + Be^{i\varepsilon t})(A + Be^{i\varepsilon t})^*} = |A + Be^{i\varepsilon t}|.$$

$$\Rightarrow \cancel{C^2 = (A+Be^{i\varepsilon t})(A+Be^{-i\varepsilon t})}$$

$$\cancel{= |A|^2 + |B|^2 + (BA^* + AB^*)(e^{i\varepsilon t} + e^{-i\varepsilon t})}$$

$$\cancel{= |A|^2 + |B|^2 + (BA^* + AB^*) \cos \varepsilon t}.$$

If we denote

$$A = a e^{i\alpha}, \quad B = b e^{i\beta}.$$

$\underset{\text{amplitude}}{a}$, $\underset{\text{phase}}{\alpha}$; $\underset{\text{amplitude}}{b}$, $\underset{\text{phase}}{\beta}$.

$$\Rightarrow c^2 = (a e^{i\alpha} + b e^{i\beta} e^{i\varepsilon t})(a e^{-i\alpha} + b e^{-i\beta} e^{-i\varepsilon t})$$

$$= a^2 + b^2 + ab\, e^{-i(\varepsilon t - \alpha + \beta)} + ab\, e^{i(\varepsilon t - \alpha + \beta)}.$$

$$= a^2 + b^2 + 2ab \cos(\varepsilon t - \alpha + \beta).$$

$$\Rightarrow C^2 \in [(a-b)^2, (a+b)^2]$$

$$\boxed{|a-b| \le C \le |a+b|}$$ with period "ε"

That's the beats phenomenon.

- Damped oscillation.

Related mathematics (Second order linear non-homogeneous equation).

Def: $F_f(\dot{x})$: frictional force.

Using Expansion find

$$F_f(\dot{x}) = F_f(0) + F_f'(0)\dot{x} + O(\dot{x}^2)$$

as $F_f(0)=0$. define. $F_f'(0) = -\alpha$.

$\Rightarrow F_f(\dot{x}) \approx -\alpha\dot{x}$

Applying on the RHS of EoM. $~~\ddot{x}+\omega^2 x =~~ m\ddot{x} = -kx$

$\Rightarrow m\ddot{x} = -kx - \alpha\dot{x} \Leftrightarrow \ddot{x} = -\frac{k}{m}x - \frac{\alpha}{m}\dot{x}$

Def: $\omega_0^2 = \frac{k}{m}$, $2\lambda = \frac{\alpha}{m}$.

ω_0: free oscillation frequency. λ: damped coefficient.

So, our EoM becomes

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0.$$

the characteristic equation is.

$$\varphi^2 + 2\lambda\varphi + \omega_0^2 = 0 \Rightarrow \begin{cases} \varphi_1 = -\lambda + \sqrt{\lambda^2 - \omega_0^2} \\ \varphi_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2} \end{cases}$$

The solution is given by,

$$x = C_1 e^{\varphi_1 t} + C_2 e^{\varphi_2 t}.$$

Here we discuss two cases.

② $\varphi_1, \varphi_2 \in \mathbb{R}$. ① $\varphi_1, \varphi_2 \in \mathbb{C}$.

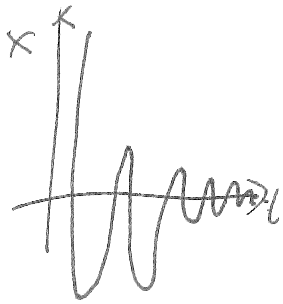
①: $\varphi_1, \varphi_2 \in \mathbb{C}$. $\Leftrightarrow \lambda < \omega_0$.

$$\begin{cases} \varphi_1 = -\lambda + i\sqrt{\omega_0^2 - \lambda^2} = -\lambda + \omega i \\ \varphi_2 = -\lambda - i\sqrt{\omega_0^2 - \lambda^2} = -\lambda - \omega i \end{cases}$$

$$\Rightarrow x = \mathrm{Re}\left\{ c_1 e^{(-\lambda + i\omega)t} + c_2 e^{(-\lambda - i\omega)t} \right\}$$

$$= \mathrm{Re}\left\{ e^{-\lambda t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t} \right) \right\}$$

$$= \mathrm{Re}\left\{ e^{-\lambda t}\left[(c_1+c_2)\cos\omega t + i(c_1-c_2)\sin\omega t \right] \right\}$$



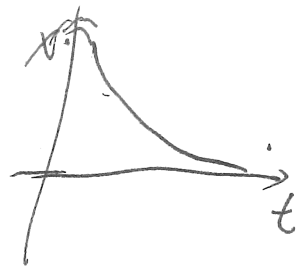
$$= e^{-\lambda t}\underbrace{(c_1+c_2)}_{A}\cos\omega t$$

$$= A \cdot \underbrace{e^{-\lambda t}}_{\text{damped term}}\cos\omega t, \quad \omega = \sqrt{\omega_0^2 - \lambda^2},$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2 e^{-2\lambda t} = E_0 e^{-2\lambda t}.$$

② when $\lambda > \omega_0$, $\varphi_1, \varphi_2 \in \mathbb{R}$.

$$x = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega_0^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega_0^2})t}.$$



③ For $\lambda = \omega_0$.

$\underline{\varphi_1 = \varphi_2 = -\lambda}$. the root is degenerate.

$x_1 = c_1 e^{-\lambda t}$, $x_2 = c_2 e^{-\lambda t}$

$$x \overset{x_1 + t x_2}{=} c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} = (c_1 + t c_2) e^{-\lambda t}$$

~~SPUM~~ PUM 101. Lecture. 5.

- Forced oscillations under friction.

we have disscussed the forced oscillation and damped oscillation. If we combine them together. we get a ~~kee~~ new problem.

$$\cancel{m\ddot{x} + k x = F(t) + F_f}$$

$$m\ddot{x} + kx = F(t) + F_f.$$

using the previous expansion, and agine $F(t) = f\cos\gamma t$.

$$F_f = -\alpha\dot{x}, \qquad F(t) = f\cos\gamma t.$$

$$\Rightarrow m\ddot{x} + kx = f\cos\gamma t - \alpha\dot{x}$$

$$\Leftrightarrow \ddot{x} + \frac{k}{m}x = \frac{f}{m}\cos\gamma t - \frac{\alpha}{m}\dot{x}$$

let $\frac{k}{m} = \omega_0^2$, $\frac{\alpha}{m} = 2\lambda$.

$$\Rightarrow \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f}{m}\cos\gamma t.$$

the solution x can be given by $x = \mathrm{Re}(y)$

wum, y satisfies.

$$\boxed{\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y = \frac{f}{m}e^{i\gamma t}.}$$

The solution is given by two parts $y_{homo} + y_{inhomo}$