- Forced oscillations under friction.

We have discussed the forced oscillation and damped oscillation. If we combine them together, we get a new problem.

~~$m\ddot{x} + kx = F(t) + F_f$~~

$$m\ddot{x} + kx = F(t) + F_f.$$

Using the previous expansion, and again $F(t) = f\cos\gamma t$:

$$F_f = -\alpha\dot{x}, \quad F(t) = f\cos\gamma t.$$

$$\Rightarrow m\ddot{x} + kx = f\cos\gamma t - \alpha\dot{x}$$

$$\Leftrightarrow \ddot{x} + \frac{k}{m}x = \frac{f}{m}\cos\gamma t - \frac{\alpha}{m}\dot{x}$$

Let $\frac{k}{m} = \omega_0^2$, $\frac{\alpha}{m} = 2\lambda$.

$$\Rightarrow \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = \frac{f}{m}\cos\gamma t.$$

The solution x can be given by $x = \mathrm{Re}[y]$ where y satisfies

$$\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y = \frac{f}{m}e^{i\gamma t}.$$

The solution is given by two parts $y_{homo} + y_{inhomo}$

Consider $\ddot{y} + 2\lambda\dot{y} + \omega_0^2 y = 0$

$\Rightarrow \varphi_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$, $y_{hom} = Ae^{(-\lambda+\sqrt{\lambda^2-\omega_0^2})t} + Be^{(-\lambda-\sqrt{\lambda^2-\omega_0^2})t}$

$= e^{-t\lambda}\left(Ae^{t\sqrt{\lambda^2-\omega_0^2}} + Be^{-t\sqrt{\lambda^2-\omega_0^2}}\right)$

For inhomogeneous solution, let $y_{inhom} = Ce^{i\gamma t}$.

$-C\gamma^2 e^{i\gamma t} + 2\lambda\gamma i C e^{i\gamma t} + \omega_0^2 C e^{i\gamma t} = \frac{f}{m} e^{i\gamma t}$

$e^{i\gamma t} \neq 0 \Rightarrow C(-\gamma^2 + 2\lambda\gamma i + \omega_0^2) = \frac{f}{m}$

$\Rightarrow C = \frac{f}{(\omega_0^2 - \gamma^2 + 2\lambda\gamma i)m}$

So, solution is given by

$y = y_{hom} + y_{inhom} = e^{-\lambda t}\left(Ae^{t\sqrt{\lambda^2-\omega_0^2}} + Be^{-t\sqrt{\lambda^2-\omega_0^2}}\right) + \frac{f}{(\omega_0^2-\gamma^2+2\lambda\gamma i)m} e^{i\gamma t}$

↓ can not be distinguished out ~~which term~~ the real part.

Notice. $\frac{f}{m(\omega_0^2-\gamma^2+2\lambda\gamma i)} = \frac{f(\omega_0^2-\gamma^2-2\lambda\gamma i)}{m[(\omega_0^2-\gamma^2)^2+4\lambda^2\gamma^2]} = b\cdot e^{i\delta}$.

$\tan\delta = \frac{-2\lambda\gamma}{\omega_0^2-\gamma^2}$, $b = \frac{f}{m\sqrt{(\omega_0^2-\gamma^2)^2+4\lambda^2\gamma^2}}$

So. $y = y_{hom} + b\cdot e^{i(\gamma t+\delta)}$.

We consider $\lambda < \omega_0$: (the other two cases we do not interested in.)

$$y = e^{-\lambda t}(Ae^{it\omega} + Be^{-it\omega}) + be^{i(\gamma t+\delta)}.$$

$$\mathrm{Re}[y] = e^{-\lambda t} a\cos(\omega t + \alpha) + b\cos(\gamma t+\delta).$$

$$\Rightarrow x = ae^{-\lambda t}\cos(\omega t + \alpha) + b\cos(\gamma t + \delta) \quad (\lambda < \omega_0).$$

① As $t \to \infty$. $x = b\cos(\gamma t + \delta)$.

② $\gamma \to \omega_0$, $b = \dfrac{f}{2\lambda\gamma m}$. (For forced oscillation not without friction, $b \to \infty$)

Resonance.

③ As f is fixed, when $\gamma = \sqrt{\omega_0^2 - 2\lambda^2}$, $b \to$ maximum.

④ Resonance area. let $\gamma = \omega_0 + \varepsilon$. ($\varepsilon \to 0$) assuming $\lambda \ll \omega_0$ then $\gamma^2 - \omega_0^2 = (\gamma+\omega_0)(\gamma-\omega_0) = 2\omega_0\varepsilon$.

and $2i\lambda\gamma \doteq 2i\lambda\omega_0$.

$$\Rightarrow b = \frac{f}{2m\omega_0\sqrt{\varepsilon^2+\lambda^2}} \qquad \tan\delta = \frac{\lambda}{\varepsilon}.$$

(For pure forced oscillation

$$x = a\cos(\omega t + \alpha) + \frac{f}{m(\omega_0^2 - \gamma^2)}\cos(\gamma t + \beta).$$
)

$$\begin{cases} \gamma \to \omega_0 \to \infty \\ 0 \leftarrow \omega \leftarrow \gamma \end{cases}$$

— dissipative function

For multi-freedom system, the frictional force

$$F_{fi} = -\sum_{k} \alpha_{ik}\,\dot{x}_k$$

Consider all coordinates system are symmetry which means $\alpha_{ik} = \alpha_{ki}$.

We could get another expression of F_{fi}:

$$F_{fi} = -\frac{\partial F}{\partial \dot{x}_i}$$

where $F = \frac{1}{2}\sum_{i,k} \alpha_{ik}\,\dot{x}_i\,\dot{x}_k$ → quadratic ~~quadratic~~ form

$$= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots \\ \alpha_{21} & \alpha_{22} & \\ \vdots & \ddots & \vdots \\ & \cdots & \alpha_{nn} \end{bmatrix}\begin{bmatrix} x_1, & x_2 & \cdots & x_n \end{bmatrix}$$

due to the symmetry of space ($\alpha_{ik} = \alpha_{ki}$)

F is called dissipative function. It describes the dissipative rate of energy in the system. When we take the time derivative to the energy ($\mathcal{L} = T - U$).

$$E = 2T - \mathcal{L}, \quad T = \frac{1}{2}\sum_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\dot{x}_i$$

~~$$\frac{dE}{dt} = 2\frac{dT}{dt} - \frac{d}{dt}\mathcal{L} = 2\frac{d(T-U)}{dt} - \frac{d}{dt}\mathcal{L}$$~~

$$\Rightarrow \frac{dE}{dt} = \frac{d}{dt}\left(\sum_i \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}\right) = \sum_i\left(\ddot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\dot{x}_i\right) - \frac{d\mathcal{L}}{dt}$$

Notice $\frac{d\mathcal{L}}{dt} = \sum_i\left(\dot{x}_i \frac{\partial \mathcal{L}}{\partial x_i} + \ddot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right)$

$$\Rightarrow \frac{dE}{dt} = \sum_i \dot{x}_i \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} \right).$$

~~By the definition. $\frac{dE}{dt} =$~~

The ~~Lang~~ L-E equation here.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i} - \frac{\partial F}{\partial \dot{x}_i}$$

quadratic

↓

$$\Rightarrow \frac{dE}{dt} = -\sum_i \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}, \quad \text{as } F = \frac{1}{2}\sum \alpha_{ik} \dot{x}_i \dot{x}_k.$$

$$\Rightarrow \frac{dE}{dt} = -2F.$$

$$\Rightarrow \frac{dE}{dt} \propto F, \quad \text{as } \overset{t\to\infty}{E \downarrow} \Rightarrow F > 0.$$

— Back to the forced frictional oscillator.

Let $-\frac{dE}{dt} = I(\gamma)$.

Then we have $I(\gamma) = 2\langle F \rangle$ → average to the oscillation period.

For 1-D oscillator. $F = \frac{1}{2}\alpha \dot{x}^2 = \lambda m \dot{x}^2$

as $t\to\infty$ $x = b\cos(\gamma t + \delta)$

$\dot{x} = \gamma b \sin(\gamma t + \delta)$.

$$\Rightarrow F = \lambda m \gamma^2 b^2 \sin^2(\gamma t + \delta)$$

take the expectation value.

$$\langle F \rangle = \lambda m \gamma^2 b^2 \langle \sin^2(\gamma t + \delta) \rangle$$

$$\langle \sin^2(\omega t + \delta) \rangle = \frac{1}{2} \quad \left(\langle \cdots \rangle = \frac{\int \gamma \sin^2(\omega t + \delta)\, dt}{\int \sin^2(\omega t+\delta)\, dt} \right).$$

5.

$\Rightarrow \langle F \rangle = \lambda m \gamma^2 b^2$. $\Rightarrow I(\gamma) = \lambda m \gamma^2 b^2$.

Normalize to be reasonable. $I(\omega) \to I(\omega_0 + \varepsilon) = I(\varepsilon)$.

$$I(\varepsilon) = \frac{f}{4m} \cdot \frac{\lambda}{\varepsilon^2 + \lambda^2}$$

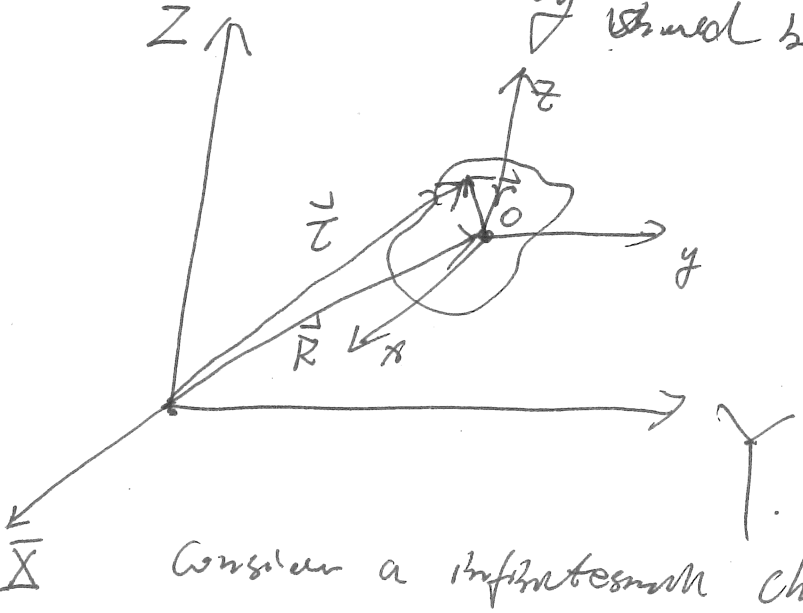
we see that $\frac{dE}{dt}$ is related to the frequency. This relation is called "dispersion"

SPUM 101 Lecture 6.

— Chapter 6. Rigid body.

— Angular velocity in (XYZ and Oxyz system)

Def: rigid body is a collection of mass points. Satisfy the distance between every two mass points is invariant. And the shape (size) changes of rigid body can be ignored. the motion of rigid body should be uniformed.



O: COM.

Consider a infinitesimal change of $\vec{r}$:

$d\vec{r} = d\vec{R} + d\vec{\varphi} \times \vec{r}$. ($\varphi$ is the angle that $\vec{r}$ rotate around O)