$\Rightarrow \langle F \rangle = \lambda m \gamma^2 b^2$. $\Rightarrow I(\gamma) = \lambda m \gamma^2 b^2$.

rem~~t~~to the reasonme. $~~I(\varepsilon)~~ I(\omega) \to I(\omega_0+\varepsilon) = I(\varepsilon)$.

$$\boxed{I(\varepsilon) = \frac{f}{4m} \cdot \frac{\lambda}{\varepsilon^2+\lambda^2}}$$

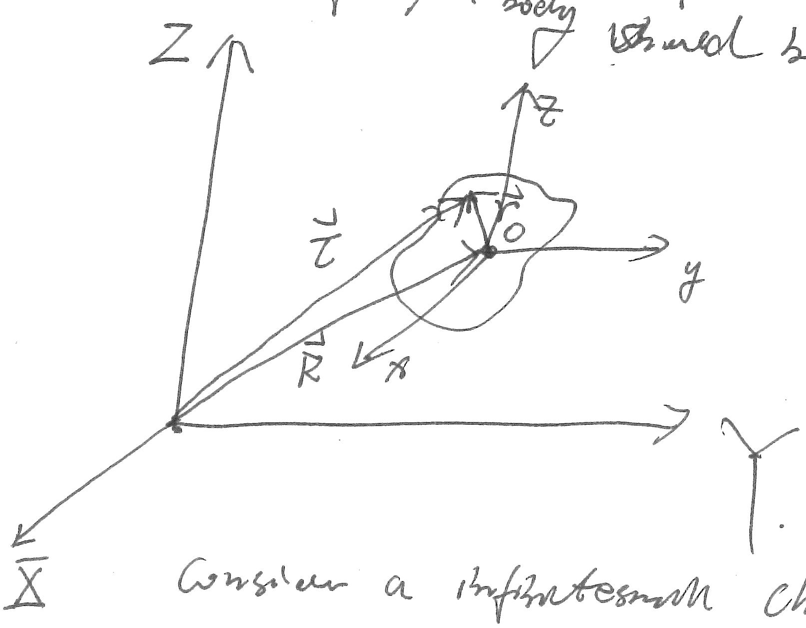
we see that $\frac{dE}{dt}$ is related to the frequency.
This ~~phenomena~~ relation is called "dispersion"

SPUM 101 Lecture ~~1~~6.

— Chapter 6. Rigid body.

— Angular velocity in (XYZ and $O\,xyz$. system)

Defs rigid body is a collection of mass ~~po~~ points. ~~with~~ ~~which~~ that satisfy the ~~disten~~ distence between ~~each~~ every two mass points is invariant. And the shape (size) can be ~~ig~~ ignored ~~neglet~~. in the motion of rigid body changes of should be ~~ango~~ uniformed. ~~and~~



O: COM.

Consider a infinitesmall change of $\vec{r}$:

$d\vec{r} = d\vec{R} + d\vec{\varphi} \times \vec{r}$. ($\varphi$ is the ange that $\vec{r}$ rotate ~~along~~ around O)

$$\Rightarrow \frac{d\vec{z}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{\varphi}}{dt} \times \vec{r}'$$

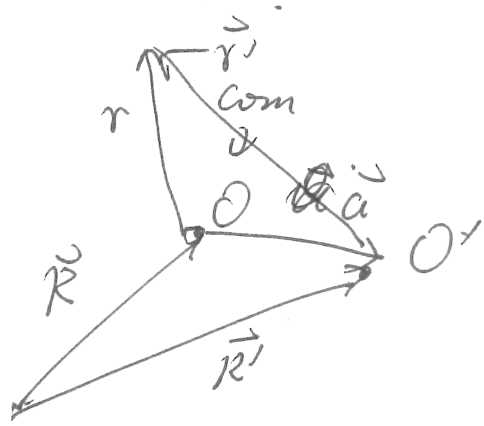
define $\vec{v} = \frac{d\vec{z}}{dt}$, $\vec{V} = \frac{d\vec{R}}{dt}$. $\vec{\Omega} = \frac{d\varphi}{dt}$.

$$\Rightarrow \boxed{\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}'}$$

total velocity

velocity of Com

rotation angular momentum.



Consider change the $\vec{O}$ point from Com to another arbitrarily point $\vec{R} + \vec{a} = \vec{R}'$

$$\Rightarrow \vec{r} = \vec{r}' + \vec{a}$$

$$\Rightarrow \vec{v} = \vec{V} + \vec{\Omega} \times (\vec{r}' + \vec{a}) = \vec{V} + \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \vec{r}'$$

$$\Rightarrow \begin{cases} \vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a} \quad ① \quad = \vec{V}' + \vec{\Omega}' \times \vec{r}' \\ \vec{\Omega}' = \vec{\Omega} \quad ② \end{cases}$$

②: $\vec{\Omega}$ is independ with the chosen coordinate sys.

① If $\vec{V} \perp \vec{\Omega}$. then, $\vec{V}' \perp \vec{\Omega}'$.

That ~~mean~~ means. we can ~~dynam~~ dynamically choose an instant center point O' s.t. $\vec{V}' = 0$.

— suffix notation. and rotational ~~ini~~ initia ~~tensor~~ tensor.

suffix notation $(i, j, k.) = (x. y. z)$

example. $a_i a_j = \begin{bmatrix} a_x a_x & a_x a_y \\ a_y a_x & a_y a_y \end{bmatrix}$ rank 2. 2x2 tensor.

~~ten~~ tensor: $\begin{cases} \text{rank 0. scalar} \\ \text{rank 1. vector} \\ \text{rank 2: matrix} \\ \text{rank 3. cubic} \end{cases}$

example 2. Kronecker delta. $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

example 3. permutation $\varepsilon_{ijk} = \begin{cases} 0 & i,j,k \text{ exist two or more equal term} \\ 1 & i,j,k \text{ are cycling order} \\ -1 & i,j,k \text{ are anti-cycling} \end{cases}$

Consider ~~the~~ a rigid body as a collection of discrete parts

$$T = \sum_i m_i \frac{v_i^2}{2} = \sum_i \frac{m_i}{2}(\vec{V} + \vec{\Omega}\times\vec{r}_i)^2 = \sum_i \frac{m_i}{2} V^2 + \sum_i m_i \vec{V}\cdot(\vec{\Omega}\times\vec{r}_i)$$

$$+ \sum_i \frac{m_i}{2}(\Omega \times r_i)^2 = \frac{V^2}{2}\sum_i m_i + \sum_i m_i \vec{V}_i \cdot (\Omega \times \vec{r}_i) + \sum_i \frac{m_i}{2}(\Omega\times r_i)^2$$

let $m = \sum_i m_i$ the second term here,

$$\sum_i m_i \vec{V}\cdot(\vec{\Omega}\times\vec{r}_i) = \cancel{(\vec{V}\times\vec{\Omega})\cdot\sum m_i \vec{r}_i} = 0 \quad - ②$$

$$\sum_i (\vec{\Omega}\times\vec{r}_i)^2 \frac{m_i}{2} = \frac{1}{2}\sum m \left[\Omega^2 r^2 - (\Omega\cdot r)^2\right] \quad - ③$$

② = 0. can be prove by

$$m_i \vec{V}_x (\vec{\Omega}\times\vec{r}_i)_x = m_i \vec{V}_x(\Omega_y r_{iz} + \Omega_z r_{iy}) \cancel{= m_i}$$

as $\sum_i m_i \vec{r}_i = \sum_i m_i (r_{ix}\hat{x} + r_{iy}\hat{y} + r_{iz}\hat{z}) = 0$

$\Rightarrow \sum_i m_i r_{ix} = 0 \ldots$

$$\Rightarrow \sum_i m_i V_x (\vec{\Omega}\times\vec{r}_i)_x = V_x\left(\Omega_y \sum_i m_i r_{iz} + \Omega_z \sum_i m_i r_{iy}\right) = 0$$

the same for the other components.

Therefore,

$$T = \frac{m}{2} V^2 + \frac{1}{2}\sum m\left[\Omega^2 r^2 - (\Omega\cdot r)^2\right]$$

$$\underbrace{\frac{m}{2}V^2}_{\text{translation}} \qquad \underbrace{\frac{1}{2}\sum m\left[\Omega^2 r^2 - (\Omega\cdot r)^2\right]}_{\text{rotation}}$$

Using ~~ten~~ suffix notation.

$$T_{rot} = \frac{1}{2}\sum_l m_l \left[\Omega_i^2 x_j^2 - \Omega_i x_i \Omega_k x_k\right]$$

Where $\Omega_i = \Omega_k \delta_{ik}$.

$$\Rightarrow T_{rot.} = \frac{1}{2}\sum_l m_l \left[\Omega_i \Omega_k \delta_{ik} x_j^2 - \Omega_i \Omega_k x_i x_k\right]$$

$$= \frac{1}{2}\Omega_i \Omega_k \sum_l m_l (x_j^2 \delta_{ik} - x_i x_k)$$

define $I_{ik} = \sum_l m_l (x_j^2 \delta_{ik} - x_i x_k)$ ← inertia tensor.

$$T = \frac{m}{2}V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k .$$

– properties

① $I_{ik} = I_{ki}$.

②

$$I_{ik} = \begin{bmatrix} \sum_l m_l (y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum myz \\ -\sum mzx & -\sum mxy & \sum m(x^2+y^2) \end{bmatrix}$$

I_{xx} I_{yy} I_{zz} called ~~trust~~ moments of inertia.

9

③ Diagonalization. eigenvalue. $I_1 . I_2 . I_3$.

For ~~an~~ can ~~?~~ chosen basis: $\vec{x}_1, \vec{x}_2, \vec{x}_3$. the $\hat{I}$ can be diagonalized.

$$\hat{I}\vec{x} = \lambda\vec{x} \Rightarrow \begin{cases}\lambda_1=I_1, & \vec{x}_1\\ \lambda_2=I_2 & \vec{x}_2 \\ \lambda_3=I_3. & \vec{x}_3\end{cases}$$

$I_1 I_2 . I_3$: principle moments of inertia

x_1, x_2, x_3. called principal axes of inertia.

solution is given by $\det(\hat{I}-\lambda \cdot I_{33})=0$

if $I_1=I_2\neq I_3$. we call it symmetrical top.

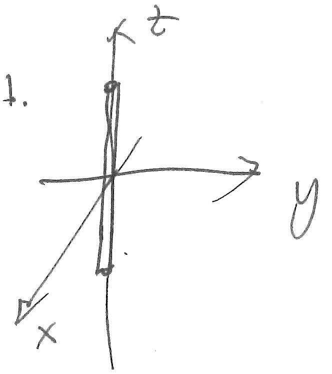
if $I_1\neq I_2\neq I_3$ ... asymmetrical top

if $I_1=I_2=I_3$ ... spherical top ($\cancel{\vec{x}_3 \perp \vec{x}_2}$ $\vec{x}_i\cdot\vec{x}_j=0$)

* For continuous cases

$$I_{ik} = \int \rho(x_l^2\delta_{ik} - x_i x_k)\, d^3r .$$

Ex1.



$$I_{xx} = \int \rho\lambda(\cancel{z})(y^2+z^2-x^2)\, d^3r .$$

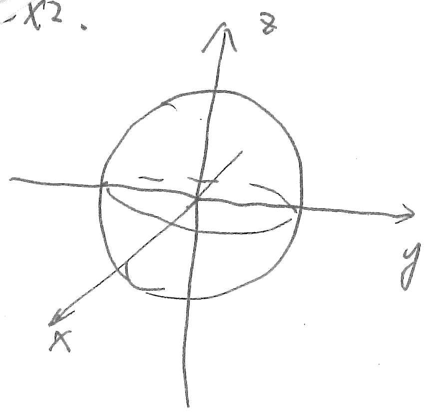
$$= \int \lambda(\cancel{z})(y^2+z^2-x^2)\, dz\, dx\, dy .$$

$$= \lambda \int_{-\frac{l}{2}}^{\frac{l}{2}} z^2 dz = \lambda \cdot \frac{2}{3}\left(\frac{l}{2}\right)^3$$

$$= \frac{\lambda l^3}{12} = \frac{ml^3}{12}$$

$I_{xx}=I_{yy}=\frac{ml^2}{12}$, $I_{zz}=0$

-X2.



$$I_{xx} = \rho \int (\overset{x^2}{y^2}+z^2-x^2)\, d^3r .$$

$$\begin{cases} x = r\cos\varphi\sin\theta . \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta . \end{cases}$$

$$d^3r = r^2\, dr\, d\varphi\, d\theta$$

$$\Rightarrow \cancel{I_{xx} = \rho \int r^4 (\sin^2\varphi\sin^2\theta - \cos^2\varphi\sin^2\theta + \cos^2\theta)\, d\varphi}\, dr\, d\theta .$$

$$\cancel{= \rho\int r^4\, dr \cdot \iint(-\sin^2\theta\cos 2\varphi + \cos^2\theta)}\, d\varphi\, d\theta .$$

$$\cancel{= \rho\, \frac{1}{5}R^5 . \int -1 \ 2\pi \cos^2\theta\, d\theta}$$

$$\cancel{= \frac{\rho}{5}R^5 . 2\pi .}\ \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta + 1\, d\theta .$$

$$\cancel{= \frac{\pi}{5}\rho R^5}\, \pi^2$$

$$I_{yy} = \rho\int (\overset{y^2}{z^2}+x^2-y^2)\, d^3r \qquad (I_{xx} = I_{yy} = I_{zz})$$

$$I_{zz} = \rho\int (\overset{z^2}{x^2}+y^2-z^2)\, d^3r$$

$$\Rightarrow 3I = \sum I_{xx} = 2\rho\int . x^2+y^2+z^2\, d^3r = 2\rho\int r^2\, d^3r$$

$$= \rho\, 2\int_0^R 4\pi r^4\, dr = 2\rho . \frac{4}{5}\pi R^5 .$$

$$\Rightarrow I = \frac{4}{3}\pi R^3 . \frac{2\rho}{5}R^2 = m\frac{2\rho}{5}R^2 .$$

— Angular momentum.

The angular momentum we define is ~~about~~ on the ~~center~~ CoM of the body

Def: $\vec{L} = \sum_l m_l \vec{r} \times \vec{v}$

where $\vec{v} = \vec{V} + (\vec{\Omega} \times \vec{r})$.

$$\Rightarrow \vec{L} = \sum_l m_l \vec{r} \times (\vec{V} + \vec{\Omega} \times \vec{r})$$

$$= \sum_l m_l (\underbrace{\vec{r} \times \vec{V}}_{0} + \vec{r} \times (\vec{\Omega} \times \vec{r}))$$

$$= \sum_l m_l \vec{r} \times (\vec{\Omega} \times \vec{r})$$

$$= \sum_l m_l [r^2 \vec{\Omega} - \vec{r}(\vec{r} \cdot \vec{\Omega})]$$

Using suffix notation

$$\vec{L}_i = \sum_l m_l (x_j^2 \Omega_i - x_i x_k \Omega_k) = \sum m (x_l^2 \Omega_k \delta_{ik} - x_i x_k \Omega_k)$$

$$= \Omega_k \sum_l m_l (x_j^2 \delta_{ik} - x_i x_k)$$

$$= I_{ik} \Omega_k .$$

— EoM. of rigid body.

~~$\mathcal{L} = \frac{1}{2} m V^2$~~ $\mathcal{L} = \frac{1}{2} m v^2 - U(R, \varphi)$ recall

~~$\mathcal{L} = \frac{1}{2}\sum m_l (V + \vec{\Omega} \times \vec{r})^2 - U(R)$~~ $\mathcal{L} = \frac{mV^2}{2} + \frac{1}{2}\sum_l m_l (x_j^2 \delta_{ik} - x_i x_k) \Omega_i \Omega_k - U$

~~$= \frac{1}{2} m V^2 + \frac{1}{2}\sum_l m_l (\vec{\Omega} \times \vec{r}) + \sum m_l \vec{V} \cdot (\vec{\Omega} \times \vec{r}) - U(R)$~~

We have two ~~rot~~ degree of freedom:

(V, R) (Ω, φ).

(1) $(\vec{V}, \vec{R})$

$$\frac{\partial \mathcal{L}}{\partial V} = mV \quad , \quad \frac{\partial \mathcal{L}}{\partial R} = -\frac{\partial U}{\partial R} = F_R.$$

(2) (Ω, φ)

$$\frac{\partial \mathcal{L}}{\partial \Omega_i} = 2 \cdot \frac{1}{2} \sum_l m_l (x_j^2 \delta_{ik} - x_i x_k) \Omega_k$$

$$= 2 \cdot \frac{1}{2} I_{ik} \Omega_k = I_{ik} \Omega_k = L_i$$

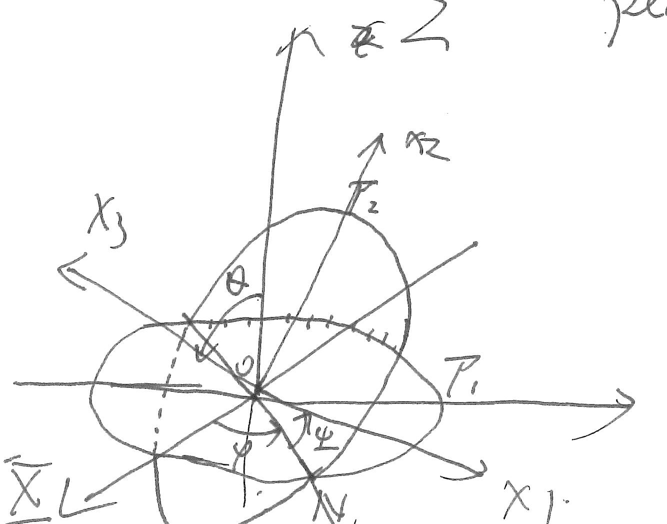
$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} = F_\varphi$$

EoM:

$$\begin{cases} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial V} = \frac{\partial \mathcal{L}}{\partial R} \Rightarrow m\dot{V} = F_R. \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \Omega} = \frac{\partial \mathcal{L}}{\partial \varphi} = I_{ik} \Omega_k = F_{\varphi i}. \end{cases}$$

— Euler's angle.

To simplify the description of rotation coordinate XYZ and x_1, x_2, x_3



plane P_1 P_2 intersect at $\vec{ON}$

(x_1, x_2, x_3) orthonormal

(X, Y, Z).

θ: $\vec{Z} \to \vec{x_3}$

φ: $\vec{X} \to \vec{ON}$

ψ: $\vec{ON} \to \vec{x_1}$

we have two ~~not~~ degree of freedom:

(V, R) (Ω, φ).

(1) $(\vec{V}, \vec{R})$

$$\frac{\partial \mathcal{L}}{\partial V} = mV \qquad , \qquad \frac{\partial \mathcal{L}}{\partial R} = -\frac{\partial U}{\partial R} = F_R.$$

(2) (Ω, φ)

$$\frac{\partial \mathcal{L}}{\partial \Omega_i} = 2 \cdot \frac{1}{2} \sum_l m_l (x_j^2 \delta_{ik} - x_i x_k) \Omega_k$$

$$= 2 \cdot \frac{1}{2} I_{ik} \Omega_k = I_{ik} \Omega_k = L_i$$

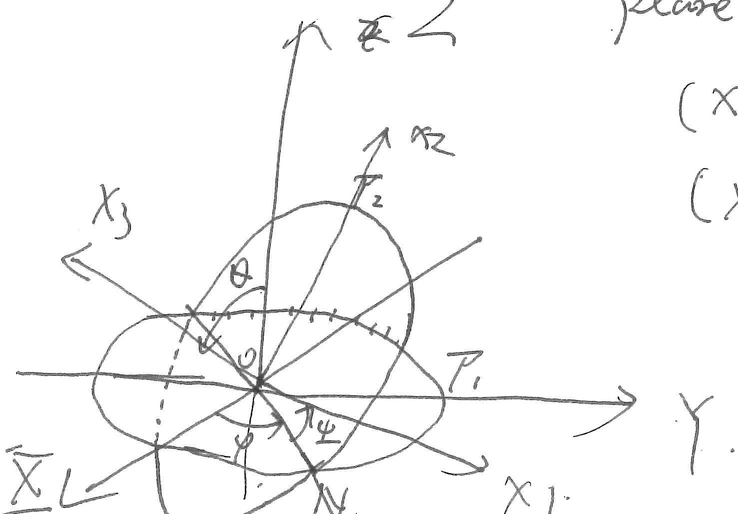
$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{\partial U}{\partial \varphi} = F_\varphi$$

EOM:

$$\begin{cases} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial V} = \frac{\partial \mathcal{L}}{\partial R} \Rightarrow m\dot{V} = F_R. \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \Omega} = \frac{\partial \mathcal{L}}{\partial \varphi} = I_{ik} \Omega_k = F_{\varphi_i}. \end{cases}$$

— Euler's angle.

To simplify the description of rotation coordinate XYZ and x_1, x_2, x_3



plane P_1 P_2 intersect at $\overrightarrow{ON}$

(x_1, x_2, x_3) orthonormal

(X, Y, Z).

θ: $\vec{Z} \to \vec{x}_3$

φ: $\vec{X} \to \overrightarrow{ON}$

ψ: $\overrightarrow{ON} \to \vec{x}_1$

$\dot\theta \parallel \vec{ON}$, $\dot\varphi \parallel \vec{Z}$, $\dot\psi \parallel \vec{x_3}$

The Euler ~~eq~~ EoM is a reregnuation of angular velocity of each x_1 x_2. x_3

$\dot\theta_1 = \dot\theta \cos\psi$. $\dot\theta_2 = -\dot\theta \sin\psi$ $\dot\theta_3 = 0$.

$\dot\varphi_1 = \dot\varphi \sin\theta \sin\psi$. $\dot\varphi_2 = \dot\varphi \sin\theta \cos\psi$. $\dot\varphi_3 = \dot\varphi \cos\theta$

$\dot\psi_1 = 0$ $\dot\psi_2 = 0$, $\dot\psi_3 = \dot\psi$

$$\Omega_i = \dot\theta_i + \dot\varphi_i + \dot\psi_i$$

$$\Rightarrow \begin{cases} \Omega_1 = \dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \Omega_2 = \dot\varphi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \Omega_3 = \dot\varphi \cos\theta + \dot\psi \end{cases}$$

with I_1. I_2. I_3. we get. ~~Trot~~

$$T_{rot} = \sum_i I_i \Omega_i$$