~~S~~PUM 101. lecture 7.

- Guage invariance of Langrangian.

consider a additional. to the langrangian.

$$\mathcal{L}' = \mathcal{L} + \frac{d}{dt} f(q,t).$$

How about the change of E-L ~~fn~~ equation?

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \ \text{①} \rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}'}{\partial q_i} = 0 \ \text{②}$$

② becomes.

$$\text{②} \Rightarrow \text{①} + \underbrace{\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left[\frac{d}{dt} f(q,t)\right] - \frac{\partial}{\partial q_i}\left[\frac{d}{dt} f(q,t)\right]}_{\text{③}} = 0.$$

From.

$$\frac{d}{dt} f(q,t) = \sum_j \frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial t}.$$

$$\overset{(1)}{\Rightarrow} \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left[\frac{d}{dt}f(q,t)\right] = \frac{d}{dt}\left(\sum_j \frac{\partial f}{\partial q_j}\delta_{ij}\right) = \frac{d}{dt}\frac{\partial f}{\partial q_i}$$

$$= \sum_j \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t}.$$

$$\overset{(2)}{} \frac{\partial}{\partial q_i}\left[\frac{d}{dt}f(q,t)\right] = \sum_j \frac{\partial^2 f}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 f}{\partial q_i \partial t}.$$

(1)-(2)

$$\Rightarrow \text{③} = 0.$$

Therefore we get

① = ②. $\mathcal{L} \to \mathcal{L}'$. the E-L equation is unchange! This property is really important. because This relation is called gauge ~~t~~ invariance. which means if we change the point of observation, reference frame the E-L Results including relative velocity is unchange.

— Maxwell's Equation, ~~Vector potential~~

In classical Electrodynamics theory the maxwell's Equation ~~in Vaccum is give~~ is given.

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} & \text{Gauss} \longrightarrow \nabla \cdot \nabla V = \nabla^2 \phi = -\dfrac{\rho}{\varepsilon_0} - \dfrac{\partial}{\partial t}\nabla \cdot \vec{A} \quad \text{possion eq.} \\ \nabla \cdot \vec{B} = 0 & \text{Gauss faraday} \longrightarrow \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial}{\partial t}\vec{B} & \text{Faraday} \longrightarrow \nabla \times \vec{E} = -\nabla \times (\nabla \phi) - \nabla \times \dfrac{\partial}{\partial t}\vec{A} = 0 - \dfrac{\partial}{\partial t}\vec{B} \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial}{\partial t}\vec{E} & \to \text{Ampère} \end{cases}$$

where $\vec{E} = -\nabla \phi - \dfrac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$

electric potential — magnetic vector potential.

~~V. Vector~~

— Vector potential.

$\vec{B} \equiv \nabla \times \vec{A}$, $\vec{A}$ unit: [V · s · m⁻¹]

$$\begin{cases} \vec{B} \\ \vec{E} \equiv -\nabla \phi \end{cases} \longrightarrow \begin{cases} \vec{B} \equiv \nabla \times \vec{A} \\ \vec{E} = -\nabla \phi - \dfrac{\partial \vec{A}}{\partial t} \end{cases}$$

Lagrangian of ~~the~~ charge particle in E-M field.

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - (e\phi - e\vec{A}\cdot\dot{\vec{q}})$$

If we do a gauge transform on the potential

$$\begin{cases} \vec{\phi} \to \\ \vec{A} \to \end{cases} \quad \begin{cases} \phi' = \phi - \frac{\partial \beta}{\partial t} \\ \vec{A}' = A + \nabla \beta \end{cases} \quad \left(\begin{array}{c} \beta \text{ is a scalar function} \\ \beta = \beta(q, t) \end{array} \right)$$

Then.

$$\mathcal{L}' = \frac{1}{2} m \dot{q}^2 - (e\phi' - e\vec{A}'\cdot\dot{\vec{q}})$$

$$= \frac{1}{2} m \dot{q}^2 - (e\phi - e\vec{A}\cdot\dot{\vec{q}}) - e\left(-\frac{\partial \beta}{\partial t} - \nabla\beta\cdot\dot{\vec{q}}\right)$$

$$= \mathcal{L} + e\left(\frac{\partial \beta}{\partial t} + \nabla\beta\cdot\dot{\vec{q}}\right)$$

$$= \mathcal{L} + e\left(\frac{\partial \beta}{\partial t} + \sum_i \frac{\partial \beta}{\partial q_i}\dot{q}_i\right)$$

$$= \mathcal{L} + \cancel{e}\, \frac{d}{dt}(e\beta)$$

$$\boxed{e\beta = \cancel{e\beta(}\, f(q, t)}$$

$\Rightarrow$ E-L equation stay un change!

— Additional info.

Gauge transform of $\vec{E}$ - $\vec{B}$ field.

Poisson Eq. $\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\frac{\rho}{\varepsilon_0}$

Consider a change of observation point.

~~$V \to V'$~~ $\phi \to \phi' = \phi + \beta$. $\vec{A} \to \vec{A}' = \vec{A} + \vec{\alpha}$

As ~~$\nabla(\nabla\times\vec{A})$~~ $\nabla\times\vec{A} = \nabla\times\vec{A}' = \vec{B} \Rightarrow \nabla\times\vec{\alpha} = 0.$

$\Rightarrow \vec{\alpha} = \nabla\lambda$ → scalar. field.

~~So, the possion Eq Eq ...~~

~~$\nabla^2\phi$~~ As ~~$\nabla\phi = \nabla\phi' = \vec{E} =$~~

$$-\nabla\phi - \frac{\partial}{\partial t}\vec{A} = -\nabla\phi' - \frac{\partial}{\partial t}\vec{A}' = \vec{E}$$

$$\Rightarrow -\nabla\phi - \nabla\beta - \frac{\partial}{\partial t}\vec{A} - \frac{\partial}{\partial t}\vec{\alpha} = \vec{E}$$

$$\Rightarrow \quad -\nabla\beta - \frac{\partial}{\partial t}\vec{\alpha} = 0 \Rightarrow \nabla\beta + \frac{\partial}{\partial t}\vec{\alpha} = 0.$$

as. $\vec{\alpha} = \nabla\lambda.$

$$\Rightarrow \nabla\underbrace{\left(\beta + \frac{\partial}{\partial t}\lambda\right)} = 0.$$

$$\Rightarrow \beta = -\frac{\partial\lambda}{\partial t}.$$

$$\Rightarrow \begin{cases} \vec{A}' = \vec{A} + \nabla\lambda. \\ \phi' = \phi - \frac{\partial\lambda}{\partial t}. \end{cases}$$

– Hamilton mechanics.

From Lagrange to Hamilton, the changes ~~is~~ are. coordinates system. $(q, \dot{q}) \to (q, p)$. , $\mathcal{L} = T - V$, $H = T + V$

~~Carti~~ ~~Car~~ Cartesian Coordinate | Canonical Coordinate.

$\mathcal{L}$: real space. (q).

phase space. (p, q).

u, we will demonstrate how to derive Hamiltonian from lagrangian. Nature.

Notice we have propose that the conservative quantity

$$\frac{d}{dt}Q = 0 \quad \text{if } Q \text{ is conserved.}$$

For a close system. $\mathcal{L}(q, \dot{q})$

$$\frac{d\mathcal{L}}{dt} = \sum\left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right) \quad - ①$$

~~If we~~ Notic we define our general momentum

E-L eq.

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \text{then,} \quad \dot{p}_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

① hence,

$$\frac{d\mathcal{L}}{dt} = \sum\left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i\right)$$

$$= \frac{d}{dt}\sum p_i \dot{q}_i$$

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{\mathcal{L} - \sum p_i \dot{q}_i}_{\text{conserved}}\Big) = 0$$

Legendre transformation.

$$\text{let } \mathcal{L} - \sum p_i \dot{q}_i = -\mathcal{H} \quad \Rightarrow \mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}. \quad - ②$$

Therefore, from ②, we get

$$\frac{\partial \mathcal{H}}{\partial q_i} = 0 - \frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i - 0 = \dot{q}_i$$

Then, we get our ~~Hamilt~~ canonical Equation of Hamilton.

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \quad ③ , \quad H = H(p, q)$$

Notice. if $H = H(p, q, t)$ we have

$$\frac{d}{dt} H = \sum_i \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right) + \frac{\partial H}{\partial t} \quad ③ \text{ in.}$$

$$= \sum_i \left(-\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} \right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}$$

$$\Rightarrow \boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

the time evolution of the system only depended on the time evolution of the Hamiltonian.

if $H = H(p, q)$

then $\boxed{\dfrac{dH}{dt} = 0}$ denote $E = H = T + V$. (total energy)

this is called "time symmetry".

— Properties of H.

(1) $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t}$

(2) $-\dfrac{\partial H}{\partial q_i} = \dot{p}_i = \dfrac{\partial L}{\partial q_i}$

(3) $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}$

(4) $\dfrac{\partial H}{\partial q_i} = 0$ if and only if $\dfrac{\partial L}{\partial q_i} = 0$

pf:

(2). $H = \sum p_i \dot{q}_i - L$.

$$\frac{dH}{dt} = \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) - \frac{dL}{dt}.$$

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$= \sum_i (\dot{p}_i \dot{q}_i + p_i \ddot{q}_i) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \boxed{\frac{dH}{dt} = - \frac{\partial L}{\partial t}.}$$

proof (3). from canonical equation

$$\dot{p}_i = - \frac{\partial H}{\partial q_i}$$

From our definition $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i$

and E-L equation $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \Rightarrow \frac{\partial L}{\partial q_i} = \dot{p}_i$

$$\Rightarrow - \frac{\partial H}{\partial q_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

proof (4). $\frac{\partial H}{\partial q_i} = 0 \Leftrightarrow \frac{\partial L}{\partial q_i} = 0.$

From (3), let $\dot{p}_i = 0. \Rightarrow - \frac{\partial H}{\partial q_i} = 0 = \frac{\partial L}{\partial q_i}.$

$$\Rightarrow - \frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial q_i} = 0.$$

$$\Rightarrow \frac{\partial H}{\partial q_i} = 0 \Leftrightarrow \frac{\partial L}{\partial q_i} = 0.$$