spum 2018 202

APAC 4004 Mathematical method of physics.

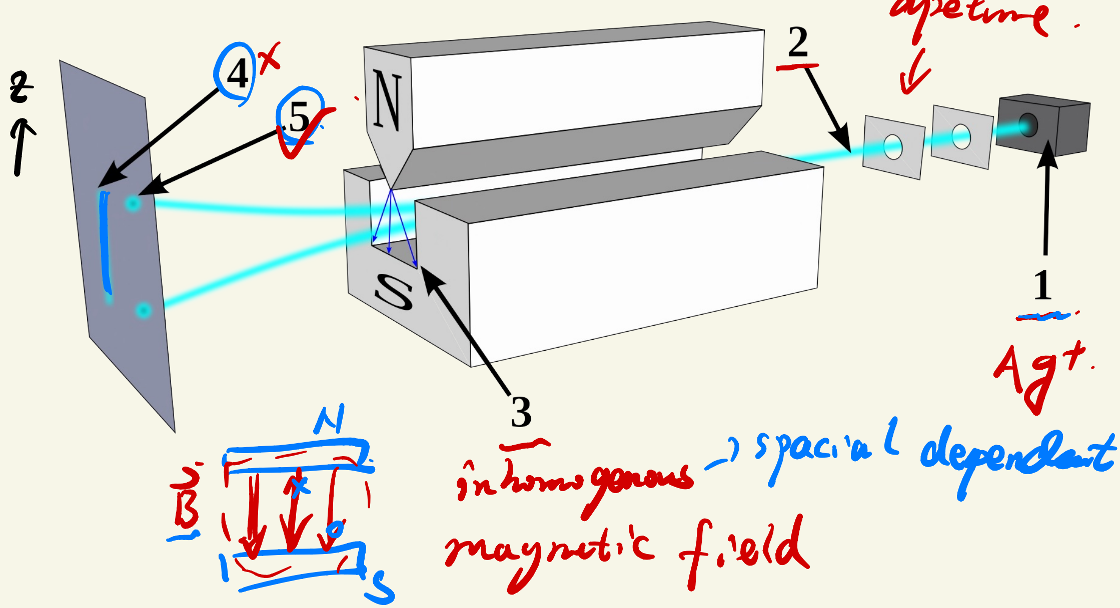
analytic solution -> numerical ~

— Stern - Gerlanch Experiment.

SG - experiment.

Ag : [Kr] 4d$^{10}$ 5s$^{1}$ → ↑ spin-up, ↓ spin-down

electron beam.




inhomogenous -> spacial dependant.

magnetic field

1922. Walter Gerlach

eletrons have two spin states


$e^-$ magnetic monment. $\vec{\mu}$

$$\vec{\mu} = -\frac{e}{m_e c} \cdot \vec{S} \rightarrow \text{spin}$$

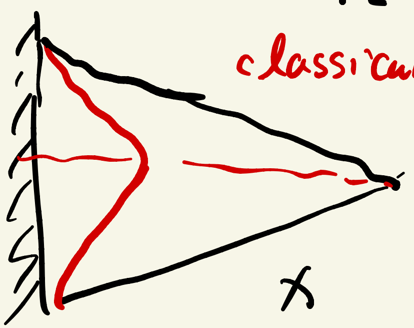
$$\vec{F}_z = \frac{\partial}{\partial z} E.$$

$$= \frac{\partial}{\partial z}\left(-\vec{\mu} \cdot \vec{B}\right)$$

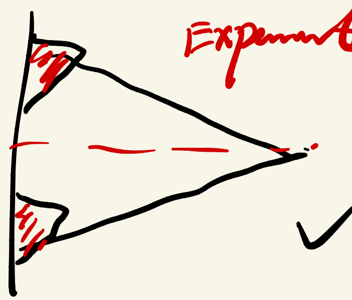
$$= \boxed{\frac{e S_z}{m_e c} \cdot \frac{\partial B_z}{\partial z}} \qquad \frac{\partial B_z}{\partial z} < 0$$

$\therefore \vec{F}_z$ oppossite to the spin of $e^-$

classically




Experimentally



conclusion: $e^-$ only has two spin states : $S_{z+}$ or $S_{z-}$.

- Fundamental principles of QM.

$\mathcal{H}$: Hirlbert space: 希尔伯特空间.

ket vector: $|a\rangle \in \mathcal{H}$.

bra vector: $\langle b| = (|b\rangle)^* \in \mathcal{H}$.

Inner product: $\langle a|b\rangle = \int a^* b \, d^3r$.

Basis: $\{|e_i\rangle\} \in \mathcal{H}$.

Complete Basis: $\forall |a\rangle \in \mathcal{H}$. $\exists c_i \in \mathbb{C}$

s.t. $|a\rangle = \sum_i c_i |e_i\rangle$.

$\propto$ operators $\hat{A}$ e.g. $\hat{A}\psi = -i\hbar \frac{\partial}{\partial x}\psi$.

observables: $\hat{H}$. $\hat{P}$. $\hat{L}$ ... $\hat{A}|\psi\rangle \begin{cases} = |\psi'\rangle \\ = a_i|\psi\rangle \end{cases}$.

unitary operator: $\hat{A}^+ = \hat{A}^{-1}$, $e^{-\frac{i\hat{H}t}{\hbar}} \cdot e^{\frac{i\hat{H}t}{\hbar}} = I$

time evolution operator.

other operators: $\hat{a}^+$, $\hat{a}$.

Fock state (number state)

$\hat{a}^+|n\rangle = \lambda|n+1\rangle$.

Example 1. $\underline{a^+ a = \hat{N}}$ (Harmonic oscillator)

$\hat{N}$: number of particles.

$\boxed{\hat{N}|n\rangle = n|n\rangle}$. phonons. $b^+ b$.
photon, coherent state.

Example 2. $\hat{H}$. observable. $|\Psi_i\rangle$ is the index

$$\underline{\hat{H}|\Psi_i\rangle = E_i|\Psi_i\rangle}.$$ Schrödinger's Eq.

Notice that. $|\Psi_i\rangle$ are eigenvectors of $\hat{H}$

— Inner product.

$\underbrace{\langle a|}_{\text{bra}}\underbrace{b\rangle}_{\text{ket}} \in \mathbb{C}$. $\boxed{\langle a|b\rangle = (\langle b|a\rangle)^*}$

* Normalization.

$|a\rangle$. $\underline{\langle a|a\rangle} \neq 1$. unnormalized.

$\langle a|a\rangle = \|a\|^2$ : $\|a\|$: norm.

$$|\bar{a}\rangle = \frac{1}{\sqrt{\langle a|a\rangle}}|a\rangle = \frac{|a\rangle}{\|a\|}$$

$$\langle\bar{a}|\bar{a}\rangle = \frac{\langle a|a\rangle}{\|a\|^2} = 1$$

- Eigenvectors as basis.

$$\hat{H}|\Psi_i\rangle = E_i|\Psi_i\rangle \quad \text{[Theorem 1.]}$$

$\{|\Psi_i\rangle\}$ is a complete basis.

Ex 3. $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2$. $\quad E_i = (n+\frac{1}{2})\hbar\omega$

$\{|0\rangle, |1\rangle, |2\rangle \cdots\}$

Ex4. central potential.

$$V(r) = A\frac{1}{r}$$


$$\hat{H}\underbrace{|n,l,jm_j\rangle}_{\text{spherical Harmonics}} = E_{nljm_j}|nljm_j\rangle$$

$\{|a_i\rangle\}$ is a set basis in $\mathcal{H}$.

For $\forall|\alpha\rangle \in \mathcal{H}$, how to describe $|\alpha\rangle$ in term of $\{|a_i\rangle\}$ ?

$$|\alpha\rangle = \underbrace{c_0}_{?}|a_0\rangle + \underbrace{c_1}_{?}|a_1\rangle + \cdots + \underbrace{c_n}_{?}|a_n\rangle$$

$$\langle a_i|a_j\rangle = \delta_{ij}$$

othogonality:

$$c_i = \langle a_i | \alpha \rangle$$

$$= \sum_{j=0}^{n} \langle a_i | c_j | a_j \rangle$$

$$= \sum_{j=0}^{n} c_j \langle a_i | a_j \rangle = \sum_{j=0}^{n} c_j \delta_{ij}$$

$$= c_i$$

so.

$$|\alpha\rangle = \sum_{i=0}^{n} c_i |a_i\rangle = \boxed{\sum_{i=0}^{n} |a_i\rangle\langle a_i|\alpha\rangle} \star$$

- Completeness.

$\{|a_i\rangle\}$ $\quad \sum_{i=0}^{n} \underline{|a_i\rangle\langle a_i|} = I \Leftrightarrow \{|a_i\rangle\}$ is complete

- projection operator.

$$\hat{A}_{a_i} \equiv |a_i\rangle\langle a_i|.$$


(1) completeness $\Leftrightarrow \sum_i \hat{A}_{a_i} = I$

(2) expansion of $\forall |\alpha\rangle$: $|\alpha\rangle = \sum_{i=0}^{n} \hat{A}_{a_i} |\alpha\rangle$

Fundamental principles

- space
  - vector, basis.
  - operators
  - inner product. $\langle a|b\rangle$.
  - superposition $|\alpha\rangle = \sum c_i |\alpha_i\rangle = \sum b_i |\beta_i\rangle$.
  - Eigenvectors
  - matrix representation.
- measurement, observables.

— Matrix representation

EX4. $\hat{H} = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix} = \Omega \hat{\sigma}_z \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$|+\rangle = [1 \;\; 0]^T$. $\quad |-\rangle = [0 \;\; 1]^T$

$$\hat{H} = \Omega |+\rangle\langle +| + (-\Omega)|-\rangle\langle -|$$

$$= \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix}$$

Let $\{|a_i\rangle\}$ are eigenvectors of $\hat{A}$ with corresponding eigenvalues $\lambda_i$. Then. $\hat{A}$ is diagonalized.

$$\boxed{\hat{A} = \sum_i \lambda_i |a_i\rangle\langle a_i|.}$$

Unluckily, $\hat{A}$ is represented under.

$$\left\{ |a_0\rangle = [1\ 0\ 0 \cdots 0]^T,\ |a_1\rangle = [0\ 1\ 0 \cdots 0]^T,\ |a_2\rangle = [0\ 0\ 1 \cdots 0]^T \cdots\cdots |a_{n-1}\rangle = [0\ 0\ 0 \cdots 1]^T \right\}$$

bits

$$\hat{A} = \sum_i \sum_j |a_i\rangle \langle a_i|\hat{A}|a_j\rangle \langle a_j| \quad \text{--- (1)}$$

$\langle a_i|\hat{A}|a_j\rangle = A_{ij}$ : matrix element.

$$(1) \Leftarrow \boxed{\hat{A} = \sum_i \sum_j A_{ij} |a_i\rangle\langle a_j|}$$

EX 5.

$$\hat{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ 4 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{|a_0\rangle} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\langle a_0|} + \cdots\cdots 4 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{|a_3\rangle} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\langle a_3|}$$

— Measurement; Observables. Uncertainty.

$(\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4}$

You can not garentee. that you probe. field will interact with your system.

* observables.

classical mechanics: $\vec{p}$. E. $\vec{L}$. : continuous.

QM. $\vec{p}$. E, $\vec{L}$, $\vec{S}$ are discrete quantities.

SG-experiment.

$\hat{A}$. is an observable, then. $\hat{A}$ is Hermitian

$$\hat{A}^{+} = (\hat{A}^{*})^{T} = \hat{A}$$

$$(\hat{A}|\Psi_i\rangle)^{+} = (a_i|\Psi_i\rangle)^{+} \quad -(2)$$

$$\langle\Psi_i|\hat{A}^{+} = \langle\Psi_i|a_i^{*} \quad -(3)$$

eigenvalues. $\in \mathbb{R}$

$$\langle\Psi_i|\hat{A} = \langle\Psi_i|a_i$$

$$\langle\vec{p}\rangle = \langle\Psi|\hat{p}|\Psi\rangle = \int \Psi^{*}\hat{p}\Psi \, d^3r$$

sandwich.

EX6. probability density of states.

$$\boxed{|\alpha\rangle = \sum_i c_i |a_i\rangle}, \quad c_i \in \mathbb{C}.$$

observe $|\alpha\rangle$. What's the probability that $|\alpha\rangle$ is at state $|a_i\rangle$

$$P(|\alpha\rangle = |a_i\rangle) = |c_i|^2 \equiv c_i^* c_i$$

$\langle \hat{A}_{a_i} \rangle$ ... projection operator.

$$\hat{A}_{a_i} = |a_i\rangle\langle a_i|.$$

$$\langle \hat{A}_{a_i} \rangle = \langle\alpha|\hat{A}_{a_i}|\alpha\rangle = \underbrace{\langle\alpha|a_i\rangle}_{c_i^*}\underbrace{\langle a_i|\alpha\rangle}_{c_i}$$

$$= c_i^* c_i = |c_i|^2.$$

* Density operator (Density matrix)

$$\boxed{\hat{\rho} \equiv |\alpha\rangle\langle\alpha| = \sum_i c_i \hat{A}_{a_i}} \quad \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}$$

A way to describe the probability distribution of the state.

## – Compatible observables.

"对易括号". Commutator.

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\{A, B\} = \sum_i \left( \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} \right)$$ Possion Braket

若 $[\hat{A}, \hat{B}] = [\hat{B}, \hat{A}] = 0$. $\hat{A}$, $\hat{B}$ are commute. "对易".

or 1) $\hat{A}$. $\hat{B}$ are compatible.

otherwise, 2) $\hat{A}$ $\hat{B}$ are incompatible

EX 7. Total Angular momentum. Hamitonian.

$$\rightarrow \mathcal{H}^{(0)}_{T} = \boxed{\frac{p^2}{2m}} - \boxed{\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}}$$

unperturbed. kentic. Central potental.

$J^2$: total angular momentum.

$[J^2, H^{(0)}] = 0$, $[J^2, L^2] = 0$

orbital angular momentum.

$[J^2, \hat{J}_z] = 0$

$H^{(0)}$ eigenstates. $\Leftrightarrow$ $J^2$ $J_z$, $L^2$ eigenstates.

addition of angular momentum.

$j(j+1)$ $m_j$ $l(l+1)$.

$$\begin{cases} J^2 |n l j m_j\rangle = j(j+1)\hbar^2 |n l j m_j\rangle \\ J_z |n l j m_j\rangle = \hbar m_j |n l j m_j\rangle \\ L^2 |n l j m_j\rangle = l(l+1)\hbar^2 |n l j m_j\rangle \\ H^{(0)} |n l j m_j\rangle = E_n |n l j m_j\rangle \end{cases}$$

separation of variables

$$\rightarrow E_n = -\frac{1}{2} M c^2 \alpha^2 Z^2 \frac{1}{n^2}$$

magnetic momentum. Fine structure constant. principle quantum number.

$$= -\frac{e^2}{2a_0} \frac{1}{n^2}$$

$\Rightarrow$ Z. Bohr's radius.