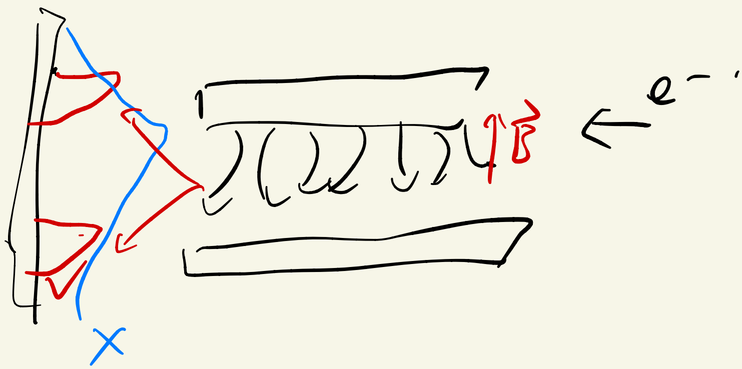
SPUM 202 Lecture 3.

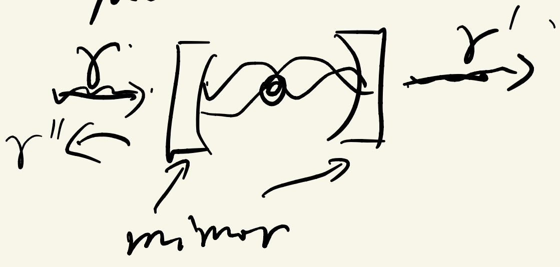
Stern - Gerlach experiment.
(SG). → the spin of electrons are discrete variables.



- $\mathcal{H}$ space : $|a\rangle$. $\hat{H}$, $\langle a|b\rangle$.
superposition. matrix representation,

- Measurement : observables. $J^2 |n\, l\, j\, m_j\rangle$
$\underline{\psi} = R_{nl}.\, Y_{ml}(\theta\phi)$

— Measurement.



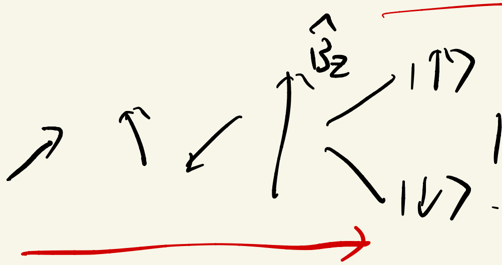
cavity QED

we first turn to the words of the great master.

A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured.

$$|\alpha\rangle \xrightarrow{\hat{P}\ \text{Measure}} \underbrace{|\alpha'\rangle}_{\hat{P}\text{'s eigenstate.}}$$

EX1. SG. $\boxed{\vec{B}\uparrow^{z}}$ $|\alpha\rangle$.



$$|\alpha\rangle \to |\hat{S}_z;+\rangle, \quad |\hat{S}_z;-\rangle$$

If $|\alpha\rangle$ is an eigenstate of our measured observables $\hat{A}$:

$$|\alpha\rangle \xrightarrow{\hat{A}} |\alpha\rangle.$$

proof:

$$\hat{A}|\alpha\rangle = a_i|\alpha\rangle.$$

$$\langle\hat{A}\rangle = \langle\alpha|\hat{A}|\alpha\rangle = \langle\alpha|a_i|\alpha\rangle$$

$$= a_i\langle\alpha|\alpha\rangle.$$

$$= a_i$$

EX2. spin 1/2 system.

$$\begin{cases} |\hat{S}_z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}. \\ |\hat{S}_z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \end{cases}$$

$$\hbar = h/2\pi$$

$$\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

$$\Rightarrow \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_z$$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Question: $|\alpha\rangle$. is a spin state

$$\langle\hat{S}_x^2\rangle \quad \langle\hat{S}_y^2\rangle \quad \langle\hat{S}_z^2\rangle \quad \langle\hat{S}^2\rangle\ ?$$

$$|\alpha\rangle = \cos\theta\,|+\rangle + \sin\theta\,|-\rangle.$$

$$\langle\alpha|\alpha\rangle = 1.$$

$$\hat{S}_x^2\,|\alpha\rangle = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|\alpha\rangle.$$

$$= \left(\frac{\hbar}{2}\right)^2 I\,|\alpha\rangle.$$

$$= \boxed{\left(\frac{\hbar}{2}\right)^2}|\alpha\rangle.$$

$$\langle S_x^2\rangle = \left(\frac{\hbar}{2}\right)^2 = \frac{1}{4}\hbar^2.$$

$$\langle S_y^2\rangle = \langle S_z^2\rangle = \frac{1}{4}\hbar^2.$$

$$\langle S^2\rangle = \langle S_x^2 + S_y^2 + S_z^2\rangle = \frac{3}{4}\hbar^2.$$

How about $\langle S_x\rangle$.

$$\hat{S}_x\,|\alpha\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\cos\theta\,|+\rangle + \sin\theta\,|-\rangle\right).$$

$$= \frac{\hbar}{2}\boxed{\left(\cos\theta\,|-\rangle + \sin\theta\,|+\rangle\right)}.$$

$$\langle\hat{S}_x\rangle = \frac{\hbar}{2}\left(\cos\theta\,\langle+| + \sin\theta\,\langle-|\right)\left(\cos\theta\,|-\rangle + \sin\theta\,|+\rangle\right).$$

$$= \frac{\hbar}{2}\,2\sin\theta\cos\theta = \frac{\hbar}{2}\sin 2\theta$$

- uncertainty. $\boxed{\Delta\hat{x}\cdot\Delta\hat{p} \geq \frac{\hbar}{2}}$

$$|\alpha\rangle \longrightarrow |\alpha'\rangle$$

- variance: 方差.

$$\langle(\Delta A)^2\rangle = \langle A\rangle_2 \equiv \langle A^2\rangle - \langle A\rangle^2$$

second cumulants.

$$= \langle(A-\langle A\rangle)^2\rangle$$

$(A-\langle A\rangle)^2$: dispersion. $=(\Delta A)^2$

$$\Delta A = A - \langle A\rangle.$$

Then variance is $\langle \Delta A^2\rangle$.

Theorem. $A, B \in \mathcal{H}$.

$$\boxed{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2}$$

This is known as the uncertainty principle.

Lemma: The Schwarz inequality.

$a, b \in \mathbb{R}^+$
$c\ d$.

$$(a^2+b^2)(c^2+d^2) \geq (ab+cd)^2$$

$|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ then,

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Lemma 2. $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$. $\hat{x}$ is an operator in $\mathcal{H}$

then,

$$\langle\alpha|\hat{x}|\beta\rangle \in \mathbb{R}.$$

if and only if $\hat{x}^+ = \hat{x}$. Hermitian.

$$\langle\alpha|\hat{x}|\beta\rangle^* = \langle\beta|\hat{x}^+|\alpha\rangle = \langle\beta|\hat{x}|\alpha\rangle$$
$$= \langle\alpha|\hat{x}|\beta\rangle$$

Lemma 3. $\langle\alpha|\hat{x}|\beta\rangle = \gamma i \quad \gamma \in \mathbb{R}$.

if and only if $\hat{x}^+ = -\hat{x}$. $i^2 = -1$.

Anti-Hermitian.

$$\Delta A \equiv A - \langle A\rangle.$$

$$\left(\overset{\langle 1|}{\langle\alpha|\Delta A^\dagger}\right)\left(\overset{|1\rangle}{\Delta A|\alpha\rangle}\right)\left(\overset{\langle 2|}{\langle\alpha|\Delta B^\dagger}\right)\cdot\left(\overset{|2\rangle}{\Delta B|\alpha\rangle}\right).$$

$$= \langle\alpha|(\Delta A)^2|\alpha\rangle\langle\alpha|(\Delta B)^2|\alpha\rangle.$$

$$= \langle\Delta A^2\rangle\langle\Delta B^2\rangle. \quad \leftarrow \text{Schwartz ineq.}$$

$$\langle 1|1\rangle\langle 2|2\rangle \geq$$

$$\geq \left|\left(\langle\alpha|\Delta A^{\dagger}\right)\left(\Delta B|\alpha\rangle\right)\right|^2.$$

$$= \underline{\left|\langle\alpha|\Delta A\Delta B|\alpha\rangle\right|^2} \quad \text{——} \quad ①$$

$$\underline{\Delta A\Delta B} = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}.$$

$$[A,B] = AB - BA, \quad \{A,B\} = AB + BA.$$

$$[A,B]^\dagger = -[A,B]$$

$$\{A,B\}^\dagger = \{A,B\}.$$

① becomes.

$$\left(\langle\Delta A\,\Delta B\rangle\right)^2 = \frac{1}{2}\underbrace{\langle[A,B]\rangle}_{\text{pure imaginary}} + \frac{1}{2}\underbrace{\langle\{A,B\}\rangle}_{\mathbb{R}}.$$

lemma 3. lemma 2.

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle \Delta A \Delta B\rangle|^2$$

$$= \boxed{\frac{1}{4}|\langle[A,B]\rangle|^2} + \frac{1}{4}|\langle\{A,B\}\rangle|^2$$

$$\geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

Ex 3. $\hat{x}$, $\hat{p}$.

$$\langle(\Delta x)^2\rangle\langle(\Delta \hat{p})^2\rangle \geq \frac{1}{4}|\langle[\hat{x},\hat{p}]\rangle|^2.$$

$$[\hat{x},\hat{p}]\psi = x\left(-i\hbar\frac{d}{dx}\psi\right) - (-i\hbar)(x\psi)'$$

$$= (-i\hbar)\left[x\psi' - \psi - x\psi'\right]$$

$$= -i\hbar(-\psi) = i\hbar\psi$$

$$\Rightarrow \sqrt{\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle} \geq \frac{1}{4}|i\hbar|^2$$

$$= \sqrt{\frac{\hbar^2}{4}}$$

— Change of basis. $[A, B] = 0$

suppose we have incompatible ($[A, B] \neq 0$). observables $\hat{A}$. $\hat{B}$, the ket space in questions can be viewed as being spaned either by the set $\{|a'\rangle\}$ or the set $\{|b'\rangle\}$.

$\hat{A}$'s eigenkets. $\hat{B}$'s eigenkets

EX4. $\{|\hat{S}_x; \pm\rangle\} \leftrightarrow \{|\hat{S}_z; \pm\rangle\}$

Theorem: Given two set of base kets $\{|a'\rangle\}$ $\{|b'\rangle\}$ both satifying orthonormality and completeness, then, $\exists\, U$ s.t.

正交性

完备性

$$|b^{(1)}\rangle = U|a^{(1)}\rangle, \quad |b^{(2)}\rangle = U|a^{(2)}\rangle.$$

$$|b^{(3)}\rangle = U|a^{(3)}\rangle \quad \cdots\cdots \quad |b^{(N)}\rangle = U|a^{(N)}\rangle$$

unitary matrix, 幺正.

where U satisfies $U^{\dagger}U = I = UU^{\dagger}$.

The schematics looks like.

$$\{|a'\rangle\} \xrightarrow{U} \{|b'\rangle\} \qquad |b'\rangle = U|a'\rangle$$

$$P_{a'} \longrightarrow P_{b'} \qquad P_{b'} = U P_{a'} U^\dagger$$

$$U^\dagger P_{b'} U = P_{a'}$$

proof.

$$U P_{a'} U^\dagger = \sum_i \sum_j I\,\underline{|a^{(i)}\rangle}\big(\langle a^{(i)}|\,I\,P_{a'}\,I\,|a^{(j)}\rangle\big)\underline{\langle a^{(j)}|}\,I.$$

$$= \sum_i \sum_j U^\dagger U\,|a^{(i)}\rangle\langle a^{(i)}|U^\dagger U P_{a'} U^\dagger U|a^{(j)}\rangle\langle a^{(j)}|U^\dagger U$$

$$= \sum_i \sum_j U\cancel{U^\dagger}\,|b^{(i)}\rangle\langle b^{(i)}|\,U P_{a'} U^\dagger\,|b^{(j)}\rangle\langle b^{(j)}|\,\cancel{U}U^\dagger$$

$\Rightarrow$

$$U P_{a'} U^\dagger = \sum_i \sum_j |b^{(i)}\rangle\langle b^{(i)}|\,U P_{a'} U^\dagger\,|b^{(j)}\rangle\langle b^{(j)}|.$$

define ☆ $\boxed{P_{b'} \equiv U P_{a'} U^\dagger}$.

$$P_{b'} = \sum_i \sum_j |b^{(i)}\rangle\langle b^{(i)}|\,P_{b'}\,|b^{(j)}\rangle\langle b^{(j)}|.$$

similarity transform.

* trace notation $tr(\hat{A})$; $\hat{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$tr(A) = 1 + 4 = 5$

$$tr(P_{a'}) = \sum_{a'} \langle a'|P_{a'}|a'\rangle.$$

As. $P_{a'} = U^\dagger P_{b'} U$.

$$\Rightarrow \quad tr(P_{a'}) = \sum_{a'} \langle a' | U^\dagger P_{b'} U | a' \rangle.$$

$$= \underline{\sum_{b'} \langle b' | P_{b'} | b' \rangle}.$$

$$= tr(P_{b'}).$$

$$\Rightarrow \quad tr(P_{a'}) = tr(P_{b'}). \; !!!$$

The following things can also be proved.

(1) $tr(XY) = tr(YX)$.

(2) $tr(U^\dagger X U) = tr(X)$.

(3) $tr(|a'\rangle\langle a''|) = \delta_{a'a''} = \begin{cases} 1 & \text{if } a' = a''. \\ 0 & \text{otherwise} \end{cases}$.

$|a'\rangle, |a''\rangle \in \{|a'\rangle\}$.

(4) $tr(|b'\rangle\langle a'|) = \langle a' | b' \rangle$.

— Diagonalization. 对角化.

$\hat{H}|\psi\rangle = E|\psi\rangle$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

EX5. $\hat{H} = \Omega \sigma_z + \cos\theta \, \sigma_x$.

$$\hat{\mathcal{H}} = \begin{bmatrix} \Omega & \cos\theta \\ \cos\theta & -\Omega \end{bmatrix}$$

basis. $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \xrightarrow{U} \boxed{?}$

$$\hat{\mathcal{H}} \cdot \begin{bmatrix} \Omega & \cos\theta \\ \cos\theta & -\Omega \end{bmatrix} \xrightarrow{U} \begin{bmatrix} \sqrt{\Omega^2+\cos^2\theta} & 0 \\ 0 & -\sqrt{\Omega^2+\cos^2\theta} \end{bmatrix}$$

what is U???

U is a rotational matrix.

$$U = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \qquad \phi \in [0, 2\pi)$$

$$U^\dagger \begin{bmatrix} \sqrt{\Omega^2+\cos^2\theta} & 0 \\ 0 & -\sqrt{\Omega^2+\cos^2\theta} \end{bmatrix} U = \begin{bmatrix} \Omega & \cos\theta \\ \cos\theta & -\Omega \end{bmatrix}$$

$\Rightarrow \quad U = \ldots$

then we have. $|e_1\rangle = U \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad |e_2\rangle = U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

— unitary Equivalent observables.

Theorem.

Consider two sets of othonormal basis. $\{|a'\rangle\}$, $\{|b'\rangle\}$ connected by the "U", we may construct a unitary transform of an operator "A": UAU^{-1}. Then, A and UAU^{-1} are said to be unitary equivalent

$$\begin{cases} A|a^{(l)}\rangle = \boxed{a^{(l)}}\,|a^{(l)}\rangle \\ (UAU^{-1})|b^{(l)}\rangle = \boxed{a^{(l)}}\,\boxed{|b^{(l)}\rangle} \end{cases}$$

where $|b^{(l)}\rangle = U|a^{(l)}\rangle$.

$$UU^{\dagger} = 1, \Rightarrow U^{-1} = U^{\dagger}$$

$$☆\begin{cases} A|a^{(l)}\rangle = a^{(l)}|a^{(l)}\rangle \\ (UAU^{\dagger})|b^{(l)}\rangle = a^{(l)}|b^{(l)}\rangle \end{cases}$$

$$p \rightarrow \boxed{\left(p - \frac{e\vec{A}}{c}\right)}$$