SPUM 202 Lecture 5.

— wave funtion (position).

$\{|x\rangle\}$ $\qquad \hat{X}|x'\rangle = x'|x'\rangle$.

eigenkets

(i) orthogonality

$$\langle x'|x''\rangle = \delta(x'-x'').$$

$$\text{(ii) } \langle x'|\alpha\rangle \doteq \int dx' \langle x'|x'\rangle\langle x'|\alpha\rangle$$

we define.

$$\psi_\alpha(x') \equiv \langle x'|\alpha\rangle.$$

this formalism, origined by Dirac, says the wave funtion in position space looks like the expansion coefficient of $|\alpha\rangle$ respect to the continuous spectrum of $\hat{x}$. $(\{|x\rangle\})$.

$$\langle x'|\alpha\rangle = \int dx' \langle x'|x'\rangle\langle x'|\alpha\rangle.$$

$$= \psi_\alpha(x') = \int dx' \delta(x'-x')\psi_\alpha(x')$$

$$= \psi_\alpha(x')$$

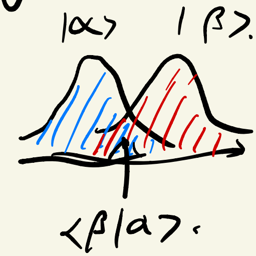
$\int dx' \delta(x'-a_0)\psi_\alpha(x')$
$= \psi_\alpha(a_0)$.

How to define the inner product of two physical states? $(|\alpha\rangle, |\beta\rangle)$.

$$\langle\beta|\alpha\rangle = \int dx' \, \langle\beta|x'\rangle\langle x'|\alpha\rangle .$$

$$= \int dx' \, \psi_\beta^*(x') \, \psi_\alpha(x'). \quad — \star$$

This can be thinked as the overlap between states $|\alpha\rangle$ and $|\beta\rangle$, or, the probability amplitude for $|\alpha\rangle$ to be fund in $|\beta\rangle$.



If $\hat{A}$ generate a set basis. $\{|a'\rangle\}$,

$$\langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle .$$

$$\Rightarrow \langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle .$$

using the language in wave mechanics.

$LHS = \psi_\alpha(x')$. $\quad RHS = \sum_{a'} c_{a'} u_{a'}(x')$,

where. $u_{a'}(x') \equiv \langle x'|a'\rangle$, $\quad c_{a'} \equiv \langle a'|\alpha\rangle$.

$\Rightarrow \psi_\alpha(x') = \sum_{a'} C_{a'} u_{a'}(x')$,

EX. $e^{ikz} \sim P_l(\cos\theta)$,
↳ associated Legendre poly

This derivation tells us.

matrix mechanics $\Leftrightarrow$ wave mechanics.

Dirac

quantum mechanics.

* matrix element. $(\hat{A})$

$$\langle\beta|\hat{A}|\alpha\rangle = \int dx' \int dx'' \langle\beta|x'\rangle \langle x'|\hat{A}|x''\rangle \langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \psi_\beta^*(x') \langle x'|\hat{A}|x''\rangle \psi_\alpha(x'')$$

Especially. if $\hat{A} = \hat{A}(x)$,

$$\Rightarrow \langle x'|\boxed{\hat{A}|x''\rangle} = A(x') \langle x'|x''\rangle$$

$$= A(x') \delta(x' - x'')$$

which reduce the formula of matrix element.

$$\langle\beta|\hat{A}(x)|\alpha\rangle = \int dx' \psi_\beta^*(x') A(x') \psi_\alpha(x'). \quad \text{— ①}$$

— Momentum operator in position space:

$$\hat{T}(dx) \longrightarrow \hat{p}, \quad \hat{p} = ?$$

Starting from the infinitesimal translation operator.

$$\hat{T}(\Delta x') \equiv 1 - \frac{i p \Delta x'}{\hbar} \quad (\Delta x \text{ is small})$$

acting on $|\alpha\rangle$

$$\Rightarrow \hat{T}(\Delta x')|\alpha\rangle = \int dx' \, \hat{T}(\Delta x')|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x'+\Delta x'\rangle\langle x'|\alpha\rangle$$

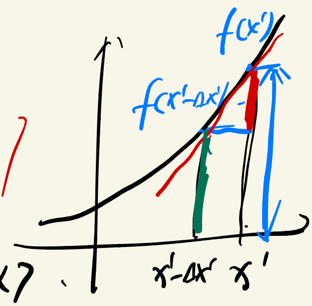
refer Lecture 4

$$= \int dx' \, |x'\rangle \underline{\langle x'-\Delta x'|\alpha\rangle}$$

where $\underline{\langle x'-\Delta x'|\alpha\rangle} = \psi_\alpha(x'-\Delta x')$

$$= \psi_\alpha(x') - \boxed{\psi'_\alpha(x')\Delta x'}$$

$$= \langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$



Bring it back to our equation, we get

$$\hat{T}(\Delta x')|\alpha\rangle = \int dx' \, |x'\rangle\left(\underline{\langle x'|\alpha\rangle} - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$\left(1 - \frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle = |\alpha\rangle - \Delta x' \int dx' \, |x'\rangle \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

$$\Rightarrow \quad \boxed{p|\alpha\rangle = \int dx' \, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\right) \langle x'|\alpha\rangle}$$

or.

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle. \quad \text{—— ②}$$

That means the momentum operator in position space is $\hat{p} = -i\hbar \frac{\partial}{\partial x'}$, $(-i\hbar \nabla')$.

Replacing $|\alpha\rangle$ by $|x''\rangle$ in eq. ②.

$$\langle x'|p|x''\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|x''\rangle$$

$$= -i\hbar \frac{\partial}{\partial x'} \delta(x' - x'').$$

The matrix element of p is. (By ①).

$$\langle \beta|p|\alpha\rangle = \int dx' \, \langle \beta|x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\right) \langle x'|\alpha\rangle.$$

$$= \int dx' \, \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'}\right) \psi_\alpha(x').$$

and

$$\langle\beta|p^n|\alpha\rangle = \int dx'\, \psi_\beta^*(x') (-i\hbar)^n \frac{\partial^2}{\partial x'^n} \psi_\alpha(x')$$

— : Momentum space wave function.

Similar to the position space.

	position space.	momentum space.
basis	$\{|x\rangle\}$.	$\{|p\rangle\}$.

Define: $p|p'\rangle = p'|p'\rangle$. Continuous spectrum.

$$\langle p'|p''\rangle = \delta(p'-p'').$$

$$|\alpha\rangle = \int dp'\, |p'\rangle\langle p'|\alpha\rangle$$

The wave function in momentum space is defined as

$$\underline{\phi_\alpha(p')} \equiv \langle p'|\alpha\rangle.$$

Normalization:

$$\langle\alpha|\alpha\rangle = \int dp' \langle\alpha|p'\rangle\langle p'|\alpha\rangle = \int dp'\, |\phi_\alpha(p')|^2$$

$$= 1.$$

Recall from equation ① in x-space.

$$\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle.$$

If we replace $|\alpha\rangle$ by $|p'\rangle$, it becomes

$$\langle x'|p|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle.$$

$$LHS = p'\langle x'|p'\rangle.$$

$$\Rightarrow -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle = p'\langle x'|p'\rangle.$$

First order, homogeneous partial differential eq-
The solution is quite simple.

$$\langle x'|p'\rangle = \underbrace{N}_{\text{normalization constant.}}\exp\left(\frac{ip'x'}{\hbar}\right)$$

If we fix x', the result implies the wavefunction of a momentum eigenstate is a plane wave.

$\exp(-ikx)$.

How to determine N?
starting from $\langle x'|x''\rangle$ in p-space.

$$\langle x'|x''\rangle = \langle x'|I|x''\rangle = \int dp'\langle x'|p'\rangle\langle p'|x''\rangle \quad ③$$

LHS $= \delta(x' - x'')$

$\langle x'|p'\rangle = N\exp\left(\frac{ip'x'}{\hbar}\right)$. $\langle p'|x''\rangle = N^* \exp\left(-\frac{ip'x''}{\hbar}\right)$.

Bring it into eq. ③.

$$\Rightarrow \delta(x'-x'') = |N|^2 \int dp' \exp\left[\frac{ip'(x'-x'')}{\hbar}\right] \quad (*)$$

This integral is a Fourier integral. Recall

$$F[\delta(x)] = \int_{-\infty}^{+\infty} \delta(x) e^{-ikx} dx$$

$$= 1$$

$$F^{-1}(F(\delta(x))) = \delta(x)$$

$$= \frac{1}{2\pi}\int 1 \cdot e^{ikx} dk$$

$$\Rightarrow \quad \delta(x) = \frac{1}{2\pi}\int e^{ikx} dk$$

if we replace k by $p'/\hbar$, we get

$$\delta(x) = \frac{1}{2\pi\hbar}\int e^{i\frac{p'x}{\hbar}} dp'$$

that gives us the final results.

$$\Rightarrow \int dp' \exp\left[\frac{ip'(x'-x'')}{\hbar}\right] = 2\pi\hbar\,\delta(x'-x'').$$

(*) becomes,

$$\delta(x'-x'') = 2\pi\hbar\,|N|^2\,\delta(x'-x'') \quad - (+)$$

$$|N|^2 = \frac{1}{2\pi\hbar}, \quad \text{take } N \in \mathbb{R}^+ ;$$

$$\Rightarrow N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\Rightarrow \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right).$$

This function has a name, called "transformation function" between x-space and p-space.

$$\langle x'|\alpha\rangle = \psi_\alpha(x') = \int dp' \boxed{\langle x'|p'\rangle} \langle p'|\alpha\rangle$$

$$= \int dp' \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right) \phi_\alpha(p').$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp'\, \phi_\alpha(p') \exp\left(\frac{ip'x'}{\hbar}\right)$$

$$\langle p'|\alpha\rangle = \phi_\alpha(p') = \int dx' \langle p'|x'\rangle \langle x'|\alpha\rangle$$

$$= \int dx' \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ip'x'}{\hbar}\right) \psi_\alpha(x')$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \psi_\alpha(x') \exp\left(-\frac{ip'x'}{\hbar}\right)$$

$$x\text{-space} \underset{F.T.^{-1}}{\overset{F.T.}{\rightleftarrows}} p\text{-space} \; ☆$$

— 3D - Case

three -D.

one-D

Replace (i) $\delta(x'-x'') \longrightarrow \delta^3(\mathbf{x}'-\mathbf{x}'')$.

(ii) $\delta(p'-p'') \longrightarrow \delta^3(\mathbf{p}'-\mathbf{p}'')$.

where $\delta^3(\mathbf{x}'-\mathbf{x}'') = \delta(x'-x'')\,\delta(y'-y'')\,\delta(z'-z'')$.

(iii) $\hat{p} = -i\hbar\frac{\partial}{\partial x'} \longrightarrow -i\hbar\nabla'$

(iv) $dx' \longrightarrow d^3x'$.

$dp' \longrightarrow d^3p'$.

Ex.

(i) $\langle \beta | p | \alpha \rangle = \int d^3x' \, \psi_\beta^*(x') (-i\hbar \nabla') \psi_\alpha(x')$.

(ii) $\langle x' | p' \rangle = \dfrac{1}{(2\pi\hbar)^{3/2}} \exp\left(\dfrac{i \vec{p}' \cdot \vec{x}'}{\hbar} \right)$.

(iii) $\psi_\alpha(x') = \dfrac{1}{(2\pi\hbar)^{3/2}} \displaystyle\int d^3p' \, \phi_\alpha(p') \exp\left(\dfrac{i p' \cdot x'}{\hbar} \right)$.

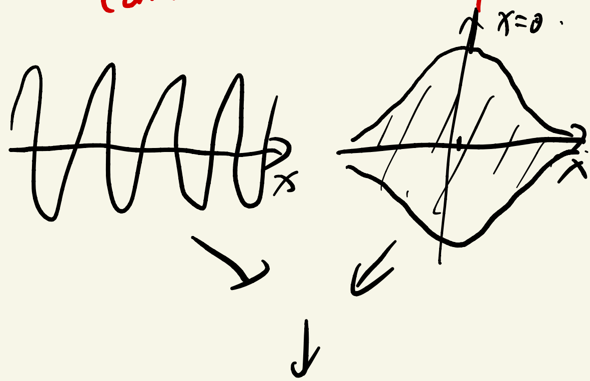
$\phi_\alpha(p') = \dfrac{1}{(2\pi\hbar)^{3/2}} \displaystyle\int d^3x' \, \psi_\alpha(x') \exp\left(-\dfrac{i p' \cdot x'}{\hbar} \right)$.

— Ex. Guassian wave packet.

$$\psi_\alpha(x') = \langle x' | \alpha \rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}} \right] \exp\left(ikx' - \frac{x'^2}{2d^2} \right).$$

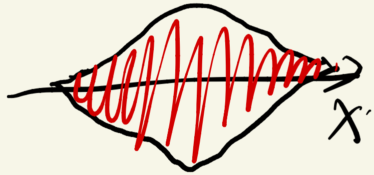
$$= \underbrace{\left[\frac{1}{\pi^{1/4}\sqrt{d}} \right]}_{\text{normalization}} \underbrace{\exp(ikx')}_{\text{carrier}} \underbrace{\exp\left(-\frac{x'^2}{2d^2} \right)}_{\text{envelope}}.$$

$\exp\left(-\dfrac{x'^2}{2d^2} \right)$ is symmetric along $x = 0$.



$\Rightarrow \langle x \rangle = 0$

$$\langle x^2 \rangle = \frac{d^2}{2}$$



$$\Rightarrow \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 .$$

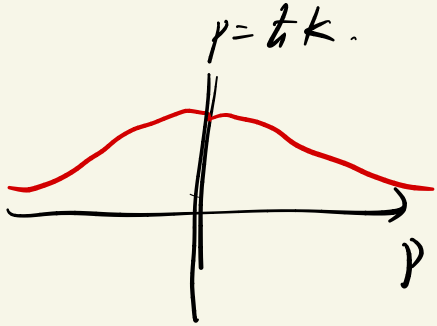
$$= \frac{d^2}{2} .$$

For momentum. $\hat{p}' = -i\hbar \frac{\partial}{\partial x'}$

$$\langle p \rangle = \int dx' \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'} \right) \langle x' | \alpha \rangle$$

$$= \hbar k .$$

$$\langle p^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$



$$\langle (\Delta p)^2 \rangle = \frac{\hbar^2}{2d^2} .$$

The uncertainty principle.

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = \frac{\hbar^2}{4}$$

the momentum wave function looks like.

$$\phi_\alpha(p') = \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2} \right]$$