SPUM 202 Lecture 6.

★ Review.

Fundamental principles of QM.

- algebraic structure.
  - Hilbert space.
  - state → $|\alpha\rangle$ ket (bra).
  - basis
  - operators.
  - superposition
- observables
  - operators. matrix.
  - eigenvalues, eigenstates.
- measurement.
  - $\langle A \rangle$.
  - uncertainty $\langle \Delta A^2 \rangle$.
- spectrum.
  - continuous / discrete. ~
  - $\hat{x}$. $\hat{p}$. canonical relation.
  - x-space ⟺ p-space.
  - $|x'\rangle \leftrightarrow |p'\rangle$.

— Quantum dynamic.

— Time evolution and the schrödinger eq.

Time-independent. S-EQ. $\mathcal{H}|\alpha\rangle = E_\alpha|\alpha\rangle$.

eigenequation.

Time-dependent. S-EQ.

$$\mathcal{H}\underbrace{|\alpha\rangle}_{|\alpha\rangle(t)} = -i\hbar\frac{\partial}{\partial t}|\alpha\rangle.$$

Notation. time: t is just a parameter in QM.

the quantum state varied with time can be represented as.

$$\underbrace{|\alpha, t_0}_{\text{index}} ; \underbrace{t}_{\text{time parameter}}\rangle$$

when $|\alpha, t_0\rangle \equiv |\alpha\rangle \doteq |\alpha, t_0; t_0\rangle$. sakurai

and $t > t_0$. consider $t \to t_0$:

$$\lim_{t \to t_0} |\alpha, t_0; t\rangle = |\alpha, t_0\rangle = |\alpha\rangle.$$

$$|\alpha, t_0\rangle \underset{\uparrow}{\rightarrow} |\alpha, t_0; t\rangle$$

time evolution.

we try to find an operator that describes this evolution process. $\mathcal{U}(t, t_0)$.

$$\mathcal{U}(t, t_0)\,|\alpha, t_0\rangle = |\alpha, t_0; t\rangle.$$

If the system energy is conserved, then:

$$\mathcal{U}^{\dagger}(t, t_0)\,\mathcal{U}(t, t_0) = I \quad -(1).$$

This is due to the symmetry of time. Although your Hamiltonian may be time-dependent. But, the property (1) can still holds in some cases.

Another property is: for $t_2 > t_1 > t_0$

$$\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\,\mathcal{U}(t_1, t_0) \quad -(2).$$

this can be visualized as

$$\begin{tikzcd} |\alpha, t_0\rangle \arrow[r] \arrow[rr, bend right, "\mathcal{U}(t_1, t_0)"'] & |\alpha, t_0, t_1\rangle \arrow[d, Leftrightarrow] \arrow[r] & |\alpha, t_0; t_1\rangle \\ & {} & \end{tikzcd}$$

Just analogue to the translation operator $\hat{T}(x)$.

Let's ~~and~~ consider an infinitesimal time evolution operator. $\mathcal{U}(t_0+dt, t_0)$.

$$\mathcal{U}(t_0+dt, t_0)|\alpha, t_0\rangle = |\alpha, t_0; t_0+dt\rangle$$

$$\lim_{dt \to 0} \mathcal{U}(t_0+dt, t_0) = 1$$

wave number. $k = \frac{\hat{p}}{\hbar}$.

The same as $\hat{T}(dx) = 1 - ikdx$, we expect $\mathcal{U}(t_0+dt, t_0)$ can be expressed as:

$$\mathcal{U}(t_0+dt, t_0) = 1 - i\Omega dt.$$

where $\Omega^\dagger = \Omega$ is a Hermitian, please prove this's convention still satisfy properties (1) and (2).

Here. $\Omega = \frac{H}{\hbar}$. $\left(k = \frac{p}{\hbar}\right)$.

$$\mathcal{U}(t_0+dt, t_0) = 1 - \frac{iHdt}{\hbar} \quad \text{— (3)}.$$

As p is the generator of $\hat{T}(dx)$, H is the generator of the $\mathcal{U}(t_0+dt, t_0)$.

— the Schrödinger Eq

$$\mathcal{U}(t+dt, t_0) = \mathcal{U}(t+dt, t)\,\mathcal{U}(t, t_0)$$

(2)

$$= \left(1 - \frac{iH\,dt}{\hbar}\right)\mathcal{U}(t, t_0)$$

(3)

$$\Rightarrow \mathcal{U}(t+dt, t_0) - \mathcal{U}(t, t_0) = -i\frac{H}{\hbar}\,dt\,\mathcal{U}(t, t_0)$$

$$\Rightarrow i\hbar\,\frac{\mathcal{U}(t+dt, t_0) - \mathcal{U}(t, t_0)}{dt} = H\,\mathcal{U}(t, t_0)$$

$$LHS = i\hbar\frac{\partial}{\partial t}\mathcal{U}(t, t_0). = H(\mathcal{U}(t, t_0)) = RHS$$

we get the S-eq of TE (time evolution operator).

apply this equation on $|\alpha\rangle = |\alpha\rangle$.

$$\boxed{i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle = H|\alpha, t_0; t\rangle} \quad (4).$$

— $\mathcal{U}(t, t_0)$

$$\hat{T}(x) = \exp\left(-\frac{i p\cdot x}{\hbar}\right)$$

$$\mathcal{U}(t, t_0) = \exp\left(-\frac{iH(t-t_0)}{\hbar}\right) \quad \text{or } t_0 = 0.$$

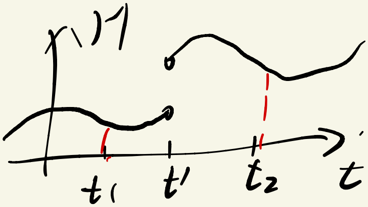
$$\mathcal{U}(t) = \boxed{\exp\left(-\frac{iHt}{\hbar}\right)} \quad \text{For } H \text{ is time-independent.}$$

phase.

If $\mathcal{H} = \mathcal{H}(t)$.

$$\mathcal{U}(t) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, \mathcal{H}(\tau)\right)$$

If $[\mathcal{H}(t_1), \mathcal{H}(t_2)] \neq 0$.



$$\mathcal{H}_0 = \frac{p^2}{2m} + V(x).$$

$$\mathcal{H}' = \frac{p^2}{2m} + V(x) - \vec{\mu}\cdot\vec{B}$$

$$\mathcal{H}(t) = \begin{cases} \mathcal{H}_0 & t < t' \\ \mathcal{H}' & t > t'. \end{cases}$$

ok, $[\mathcal{H}(t_1), \mathcal{H}(t_2)] \neq 0$. $\boxed{\text{Dyson series.}}$

— Energy eigenkets

In order to know the effect of the time evolution operator on a given state $|\alpha\rangle$, we need to expand $|\alpha\rangle$ using the energy eigenkets:

$$\mathcal{H}|a'\rangle = E_{a'}|a'\rangle.$$

where $|a'\rangle$ is the eigenstate of $\hat{A}$ and

$[A, H] = 0$. Now, expand the TE ($t_0 = 0$).

$$\exp\left(\frac{-iHt}{\hbar}\right) = \sum_{a'}\sum_{a''} |a''\rangle\langle a''|\exp\left(\frac{-iHt}{\hbar}\right)|a'\rangle\langle a'|$$

As H is diagonalized under $\{|a'\rangle\}$.

$$\Rightarrow \langle a''|\exp\left(-\frac{iHt}{\hbar}\right)|a'\rangle = \delta_{a''a'}\exp\left(-\frac{iE_{a'}t}{\hbar}\right).$$

matrix elements

$$\Rightarrow \exp\left(-\frac{iHt}{\hbar}\right) = \sum_{a'} |a'\rangle \exp\left(-\frac{iE_{a'}t}{\hbar}\right)\langle a'| \quad -(5)$$

Also, $|\alpha\rangle = \sum_{a'} C_{a'}|a'\rangle$.

$$= \sum_{a'} |a'\rangle\langle a'|\alpha\rangle.$$

we have.

$$\mathcal{U}(t)|\alpha\rangle = |\alpha; t\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle\exp\left(-\frac{iE_{a'}t}{\hbar}\right)$$

$$= \sum_{a'} C_{a'}(t)|a'\rangle.$$

$\mathcal{U}(t,0)$ changes: $C_{a'}(t=0) \to C_{a'}(t)$.

$$= C_{a'}\exp\left(-\frac{iE_{a'}t}{\hbar}\right).$$

Moreover, if $|\alpha\rangle = |a'\rangle$

$$|\alpha;t\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|a'\rangle$$

$$= \underbrace{\exp\left(-\frac{iE_{a'}t}{\hbar}\right)}_{\text{extra phase}}|a'\rangle.$$

$\phi \quad \phi' \quad \phi''$

$\uparrow \quad \downarrow \quad \uparrow$

$$|\psi\rangle = e^{i\phi}|q_1\rangle + e^{i\phi''}|q_3\rangle + e^{i\phi'}|q_2\rangle.$$

— The dependence of the expectation value.

$[A, H] = 0$, $H|a'\rangle = E_{a'}|a'\rangle$.

suppose the system start from an eigenstate $|a'\rangle$ undergoing a time evolution $\mathcal{U}(t)|a'\rangle$

$= \exp\left(-\frac{iE_{a}t}{\hbar}\right)|a'\rangle$. For an observable B, $[B, A] \neq 0$, we have the exp-value.

$$\langle B\rangle_t = \langle a'|\mathcal{U}^\dagger(t)\, B\, \mathcal{U}(t)|a'\rangle.$$

$$= \langle a'|B|a'\rangle = \langle B\rangle.$$

$\langle B\rangle$ remain constant. $\frac{d}{dt}\langle B\rangle = 0$.

For $|\alpha\rangle = \sum_{a'} C_{a'}|a'\rangle$; the results is more complicated.

$$\langle B\rangle_t = \left[\sum_{a'} c_{a'}^* \langle a'| \exp\left(\frac{iE_{a'}t}{\hbar}\right)\right] B \left[\sum_{a''} c_{a''} \exp\left(-\frac{iE_{a''}t}{\hbar}\right) |a''\rangle\right]$$

$$= \sum_{a'}\sum_{a''} c_{a'}^* c_{a''} \langle a'|B|a''\rangle \underbrace{\exp\left[\frac{-i(E_{a''}-E_{a'})t}{\hbar}\right]}_{\text{osilating term}}$$

with freqency. $\omega_{a'a''} = \frac{E_{a''}-E_{a'}}{\hbar}$.

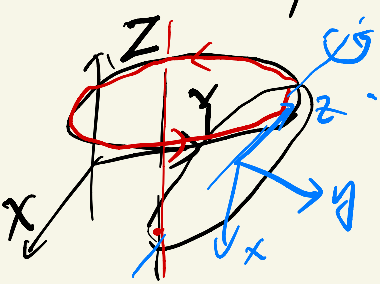
This sultion says $|\alpha\rangle$ is a non stationary state. $\exp(-i\omega t)$

but $|a'\rangle$ is a stationary state.

If $\langle a'|B|a''\rangle = \delta_{a'a''} B_{a'}$.

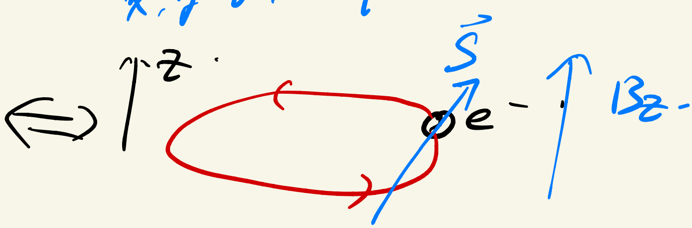
then, $\langle B\rangle = \sum_{a'} \underbrace{|c_{a'}|^2}_{P_{a'}} \underbrace{B_{a'}}_{\text{vale}} \underbrace{\exp(-i\omega_{a'a'}t)}_{\text{oscilation}}$.

Example. spin - pression



XYZ, pression.

x, y, z. spin.



$$\mathcal{H} = -\vec{\mu}_s \cdot \vec{B}$$

$S \cdot B_z$

$$= -\vec{\mu}_e \cdot \vec{B}_z = -\frac{eB_z}{m_e c} S_z \,, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$e = -1.6 \times 10^{-19}\,C$, $e<0$.

As $S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$.

where $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\because [\mathcal{H}, S_z] = 0 \Rightarrow \mathcal{H}|\pm\rangle = \mp\left(\frac{\hbar e B_z}{2 m_e c}\right)|\pm\rangle$

Define

$$\omega = \frac{|e| B_z}{m_e c} = \frac{-e B_z}{m_e c}$$

$\Rightarrow \mathcal{H}|\pm\rangle = \pm \frac{\hbar \omega}{2} |\pm\rangle$, $\mathcal{H} = \begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & -\frac{\hbar\omega}{2} \end{pmatrix}$

the TE of such $\mathcal{H}$ is $= \hbar \omega S_z$.

$$\mathcal{U}(t) = \exp\left(-\frac{i \omega S_z t}{\hbar}\right)$$

Consider a initial state $|\alpha\rangle = C_+ |+\rangle + C_- |-\rangle$.

Applying $\mathcal{U}(t)$ on it. we get

$$\mathcal{U}(t)|\alpha\rangle = C_+ \exp\left(\frac{-i\omega t}{2}\right)|+\rangle + C_- \exp\left(\frac{i\omega t}{2}\right)|-\rangle$$

Let $C_+ = C_- = \frac{1}{\sqrt{2}}$, consider $\langle S_x \rangle$. $\langle S_y \rangle$. $\langle S_z \rangle$

Ex. $\langle S_x \rangle. = \langle \alpha ; t | S_x | \alpha ; t \rangle$

$$= \underline{\langle \alpha | \mathcal{U}^{\dagger}(t)} \, S_x \, \underline{\mathcal{U}(t) | \alpha \rangle}.$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(\exp\left(\frac{i\omega t}{2}\right) \langle + | + \exp\left(\frac{-i\omega t}{2}\right) \langle - | \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}$$

$$\left(\exp\left(-\frac{i\omega t}{2}\right) | + \rangle + \exp\left(\frac{i\omega t}{2}\right) | - \rangle \right).$$

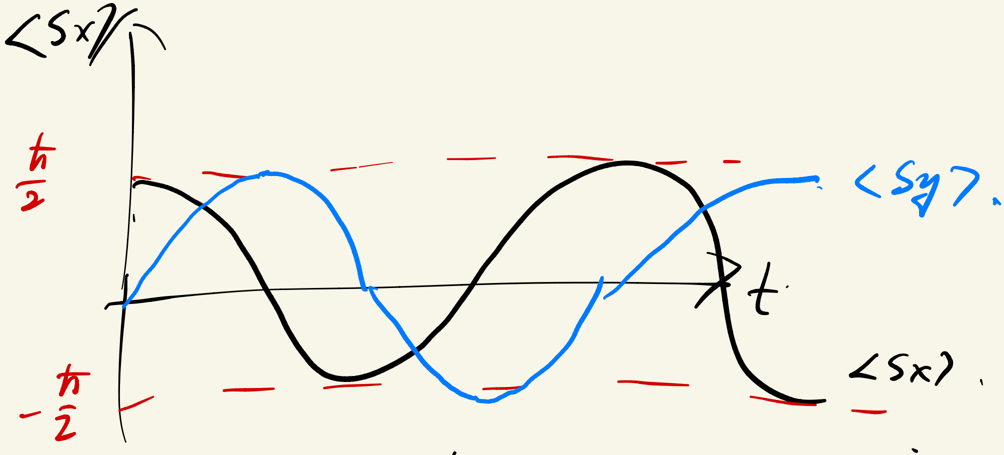
$\boxed{S_x | \pm \rangle = | \mp \rangle.}$ $\quad \langle \pm | \mp \rangle = 0.$

$$\Rightarrow \langle S_x \rangle = \frac{\hbar}{2} \left(\frac{1}{\sqrt{2}}\right)^2 \left(\underline{\exp\left(\frac{i\omega t}{2}\right) \langle + |} + \underline{\exp\left(\frac{-i\omega t}{2}\right) \langle - |} \right).$$

$$\cdot \left(\underline{\exp\left(-\frac{i\omega t}{2}\right) | - \rangle} + \underline{\exp\left(\frac{i\omega t}{2}\right) | + \rangle} \right)$$

$$= \frac{\hbar}{4} \left(\underbrace{\exp(i\omega t) + \exp(-i\omega t)}_{2\cos\omega t} \right)$$

$$= \frac{\hbar}{2} \cos \omega t$$



Also, $\langle S_y \rangle = \frac{\hbar}{2} \sin \omega t$.

$\langle S_z \rangle = 0$.